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p-ADIC PERIODS AND DERIVED DE RHAM COHOMOLOGY

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Introduction

For a smooth variety X over a base field of characteristic 0 we have its algebraic de Rham cohomology $H^{\cdot}_{\mathrm{dR}}(X) := H^{\cdot}(X_{\mathrm{Zar}}, \Omega^{\cdot}_X)$; for nonsmooth X, one defines $H^{\cdot}_{\mathrm{dR}}(X)$ using cohomological descent as in Deligne [D]. If the base field is $\mathbb{C}$, then one has the Betti cohomology $H^{\cdot}_{\mathrm{B}}(X) := H^{\cdot}(X_{\mathrm{cl}}, \mathbb{Q})$ and a canonical period isomorphism ("integration of algebraic differential forms over topological cycles")

$$\rho : H^{\cdot}_{\mathrm{dR}}(X) \xrightarrow{\sim} H^{\cdot}_{\mathrm{B}}(X) \otimes \mathbb{C} \tag{0.1}$$

compatible with the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-conjugation. To define ρ, consider the analytic de Rham cohomology $H^{\cdot}_{\mathrm{dR}}(X_{\mathrm{an}})$. There are evident maps

$$H^{\cdot}_{\mathrm{dR}}(X) \xrightarrow{\alpha} H^{\cdot}_{\mathrm{dR}}(X_{\mathrm{an}}) \xleftarrow{\beta} H^{\cdot}_{\mathrm{B}}(X) \otimes \mathbb{C}. \tag{0.2}$$

Then β is an isomorphism due to the Poincaré lemma, and $\rho := \beta^{-1}\alpha$ (the fact that ρ is an isomorphism was established by Grothendieck [Gr]).

Suppose our base field is an algebraic closure $\bar{K}$ of a p-adic field K (say, $K = \mathbb{Q}_p$). The role of $H^{\cdot}_{\mathrm{B}}(X)$ is now played by the p-adic étale cohomology $H^{\cdot}_{\mathrm{\acute{e}t}}(X, \mathbb{Q}_p)$, and Fontaine conjectured in [F1][1] the existence of a natural p-adic period isomorphism

$$\rho : H^{\cdot}_{\mathrm{dR}}(X) \otimes_{\bar{K}} \mathrm{B}_{\mathrm{dR}} \xrightarrow{\sim} H^{\cdot}_{\mathrm{\acute{e}t}}(X, \mathbb{Q}_p) \otimes \mathrm{B}_{\mathrm{dR}}. \tag{0.3}$$

Here B_{dR} is Fontaine's p-adic periods field ([F1], [F3]). Recall that it is a complete discretely-valued field whose ring of integers $\mathrm{B}^+_{\mathrm{dR}}$ contains $\bar{K}$, the residue field $\mathrm{B}^+_{\mathrm{dR}}/\mathfrak{m}_{\mathrm{dR}}$ is Tate's field $\mathbb{C}_p$, the cotangent line $\mathfrak{m}_{\mathrm{dR}}/\mathfrak{m}^2_{\mathrm{dR}}$ is the Tate twist $\mathbb{C}_p(1)$. Both sides of (0.3) carry natural filtrations (coming from the filtration of B_{dR} by powers of $\mathfrak{m}_{\mathrm{dR}}$ and the Hodge-Deligne filtration on $H^{\cdot}_{\mathrm{dR}}(X)$), and ρ is compatible with them and with the $\mathrm{Gal}(\bar{K}/K)$-conjugation. Moreover, as was envisioned by Fontaine and Jannsen ([F4], [J]), the matrix coefficients of ρ lie in the subring $\bar{K}\mathrm{B}_{\mathrm{st}}$ of B_{dR},[2] and ρ is compatible with the extra symmetries of log crystalline cohomology.

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[1]The assumption of loc. cit. that X is proper and smooth is redundant.

[2]This assertion is true for a general reason, since, by Berger's theorem [Ber], de Rham Galois modules are potentially log crystalline.

The p-adic period map was defined in three different ways in works of, respectively, Faltings, Niziol, and Tsuji (with prior crucial input of Bloch, Fontaine, Hyodo, Kato, Kurihara, and Messing; the nonproper setting was treated by Yamashita [Y]), see [Fa1], [Fa2], [N1], [N2], [Ts1], [Ts2]; the three ρ's coincide by [N3].

In the article we give another construction of ρ which is fairly direct and has the same flavor as the classical picture (0.2). The tools are derived de Rham cohomology of Illusie [Ill2, Ch. VIII] and de Jong's alterations. The companion paper [B] treats the Fontaine-Jannsen side of the story; another approach was developed by Bhatt [Bh2]. It would be very interesting to see if these methods can help to understand the Riemann-Hilbert correspondence in the p-adic setting.

An outline of the construction: First we realize $\mathrm{B}^+_{\mathrm{dR}}$ as the ring of de Rham p-adic constants in the sense of *derived* algebraic geometry. Namely, let A_{dR} be the derived de Rham algebra $L\Omega^{\cdot}_{O_{\bar K}/O_K}$ completed with respect to the Hodge filtration $F^{\cdot}$; see [Ill2, Ch. VIII (2.1.3.3)]. Here O_K, $O_{\bar K}$ are the rings of integers in K, $\bar K$. Now $\mathrm{B}^+_{\mathrm{dR}}$ identifies canonically with $\mathrm{A}_{\mathrm{dR}}\widehat{\otimes}\mathbb{Q}_p$, where $\widehat{\otimes}$ is the derived completed tensor product, so that $\mathfrak{m}^i_{\mathrm{dR}} \xrightarrow{\sim} F^i\mathrm{A}_{\mathrm{dR}}\widehat{\otimes}\mathbb{Q}_p$. This fact was observed independently by Fargues [Far].

Let $\mathcal{V}ar_F$ be the category of varieties over a field F, and $\mathcal{V}ar_F^{\mathrm{nc}}$ be the category of regular F-varieties U equipped with a regular compactification $\bar U$ with normal crossings divisor at infinity. As follows from de Jong's theorem [dJ1], the forgetful functor $\mathcal{V}ar_F^{\mathrm{nc}} \to \mathcal{V}ar_F$, $(U,\bar U)\mapsto U$, makes $\mathcal{V}ar_F^{\mathrm{nc}}$ a base for the h-topology on $\mathcal{V}ar_F$, so h-sheaves on $\mathcal{V}ar_F$ are the same as sheaves on $\mathcal{V}ar_F^{\mathrm{nc}}$ for the induced topology. For $F=\bar K$ as above, there is a finer category $\mathcal{V}ar_{\bar K}^{\mathrm{ss}}$ of ss-*pairs* $(V,\bar V)$, i.e., smooth $\bar K$-varieties V equipped with a semi-stable compactification $\bar V$ (that includes compactification in the arithmetic direction). Again by de Jong [dJ1], $\mathcal{V}ar_{\bar K}^{\mathrm{ss}}$ is a base for the h-topology on $\mathcal{V}ar_{\bar K}$.

Consider the presheaf on $\mathcal{V}ar_{\bar K}^{\mathrm{ss}}$ which assigns to $(V,\bar V)$ the derived de Rham algebra with log singularities $R\Gamma(\bar V, L\Omega^{\cdot}_{(V,\bar V)/O_K}{}^{\hat{}})$ (see [Ol]). Its h-sheafification $\mathcal{A}^\natural_{\mathrm{dR}}$ is an h-sheaf of filtered dg algebras on $\mathcal{V}ar_{\bar K}$ that contains the constant subsheaf A_{dR}. The key *p-adic Poincaré lemma* says that *the map $A_{dR}\otimes^L\mathbb{Z}/p^n\to\mathcal{A}^\natural_{dR}\otimes^L\mathbb{Z}/p^n$ is a filtered quasi-isomorphism.* It comes from the next assertion: The h-sheafification of the presheaf $(V,\bar V)\mapsto H^b(\bar V,\Omega^a_{(V,\bar V)/O_{\bar K}})$, where $\Omega^a_{(V,\bar V)/O_{\bar K}}$ is the usual locally free $\mathcal{O}_{\bar V}$-module of forms with log singularities, is an h-sheaf of $\mathbb{Q}$-vector spaces for $(a,b)\neq(0,0)$. The case $a=0$ is essentially theorem 8.0.1 from Bhatt's thesis [Bh1]; the general result is obtained by a similar method (which uses coverings of families of stable curves that come from the multiplication by p isogeny of the generalized Jacobians).

Set $R\Gamma^\natural_{\mathrm{dR}}(X):=R\Gamma(X_{\mathrm{h}},\mathcal{A}^\natural_{\mathrm{dR}})$; this is the *arithmetic de Rham complex* of X. By the above, $H^{\cdot}(R\Gamma^\natural_{\mathrm{dR}}(X)\widehat{\otimes}\mathbb{Q}_p)$ is a $\mathrm{B}^+_{\mathrm{dR}}$-algebra. One has a diagram

$$H^{\cdot}_{\mathrm{dR}}(X)\xrightarrow{\alpha}H^{\cdot}(R\Gamma^\natural_{\mathrm{dR}}(X)\widehat{\otimes}\mathbb{Q}_p)\xleftarrow{\beta}H^{\cdot}_{\text{ét}}(X,\mathbb{Q}_p)\otimes\mathrm{B}^+_{\mathrm{dR}},\tag{0.4}$$

where α is the composition $H^{\cdot}_{\mathrm{dR}}(X)\xrightarrow{\sim}H^{\cdot}R\Gamma^\natural_{\mathrm{dR}}(X)\otimes\mathbb{Q}\to H^{\cdot}(R\Gamma^\natural_{\mathrm{dR}}(X)\widehat{\otimes}\mathbb{Q}_p)$ and β is the $\mathrm{B}^+_{\mathrm{dR}}$-linear extension of the evident map (which comes from the embeddings $\mathbb{Z}/p^n\to\mathcal{A}^\natural_{\mathrm{dR}}\otimes^L\mathbb{Z}/p^n$ and the fact that the h-topology is stronger than the étale one). Since the étale and h-cohomology with torsion coefficients coincide, the Poincaré lemma implies that β is an isomorphism. Now the p-adic period map ρ is the B_{dR}-linear extension of $\beta^{-1}\alpha$. An explicit computation for $X = \mathbb{G}_m$ followed by usual tricks of the trade shows that ρ is a filtered isomorphism.

1. A derived de Rham construction of B_{dR}

1.1. *The derived p-adic completion.* Throughout the article we use (not too heavily) E_∞ algebras, for which we refer to, say, [HS].[3] Recall that E_∞ algebras are dg algebras whose product is commutative and associative up to coherent higher homotopies (more formally, E_∞ algebras are dg algebras for a resolution of the commutative algebra operad). A key fact: for any commutative (more generally, E_∞) cosimplicial dg algebra the corresponding total complex is naturally an E_∞ algebra. Thus the homotopy limit of a diagram of E_∞ algebras is an E_∞ algebra.

For a projective system of complexes of abelian groups $\ldots \xrightarrow{\phi_2} C_2 \xrightarrow{\phi_1} C_1$, one has $\mathrm{holim}\, C_n = \mathcal{C}one(\mathrm{id} - \phi : \Pi C_n \to \Pi C_n)[-1]$, where $\phi((c_n)) = (\phi_n(c_{n+1}))$. There is an embedding $\varprojlim C_n = \mathrm{Ker}(\mathrm{id} - \phi) \hookrightarrow \mathrm{holim}\, C_n$. If all ϕ_n's are surjective, then $\mathrm{id} - \phi$ is surjective; hence $\hookrightarrow$ is a quasi-isomorphism. So holim, being an exact functor, is the right derived functor of $\varprojlim$.

If $C.$ is a projective system of dg algebras, then $\mathrm{holim}\, C_n$ is naturally a dg algebra (and the above embedding is an embedding of algebras); if the C_n are commutative (or, more generally, E_∞) algebras, then $\mathrm{holim}\, C_n$ is an E_∞ algebra.

Let p be a prime. Consider the projective system of commutative dg algebras $C_n := \mathcal{C}one(\mathbb{Z} \xrightarrow{p^n} \mathbb{Z})$. It is a resolution of the projective system $\ldots \to \mathbb{Z}/p^2 \to \mathbb{Z}/p$, so $\mathbb{Z}_p^\flat := \mathrm{holim}\, C_n$ is an E_∞ algebra with $H^0\mathbb{Z}_p^\flat = \mathbb{Z}_p$ and acyclic in nonzero degrees. Set $\mathbb{Q}_p^\flat := \mathbb{Z}_p^\flat \otimes \mathbb{Q}$. For any complex F of abelian groups set

$$F\widehat{\otimes}\mathbb{Z}_p := \mathrm{holim}\,(F \otimes C_n), \quad F\widehat{\otimes}\mathbb{Q}_p := (F\widehat{\otimes}\mathbb{Z}_p) \otimes \mathbb{Q}. \tag{1.1.1}$$

These are dg $\mathbb{Z}_p^\flat$ and $\mathbb{Q}_p^\flat$-modules, so their cohomology groups are $\mathbb{Z}_p$- and $\mathbb{Q}_p$-modules, and $F \mapsto F\widehat{\otimes}\mathbb{Z}_p, F\widehat{\otimes}\mathbb{Q}_p$ are exact functors. If F is an (E_∞) dg algebra, then so are $F\widehat{\otimes}\mathbb{Z}_p$ and $F\widehat{\otimes}\mathbb{Q}_p$.

Remark. One has an evident projective system $F_{p^n}[1] \to F \otimes C_n \to F/p^nF$ of exact triangles; applying holim, we get a canonical exact triangle $\mathrm{holim} F_{p^n}[1] \to F\widehat{\otimes}\mathbb{Z}_p \to \mathrm{holim}(F/p^nF)$. Let $\hat{F} := \varprojlim F/p^nF$ be the p-adic completion of F and $T_pF := \varprojlim F_{p^n}$ be the Tate module of F. By above, we have a quasi-isomorphism $\hat{F} \xrightarrow{\sim} \mathrm{holim}(F/p^nF)$. Thus if F has no p-torsion, then $F\widehat{\otimes}\mathbb{Z}_p \xrightarrow{\sim} \hat{F}$. Similarly, if all components of F are p-divisible, then one has quasi-isomorphisms $T_pF \xrightarrow{\sim} \mathrm{holim} F_{p^n}$ and $T_pF[1] \xrightarrow{\sim} F\widehat{\otimes}\mathbb{Z}_p$. We see that $\cdot\widehat{\otimes}\mathbb{Z}_p$ is the left derived functor of the p-adic completion functor and the right derived functor of $T_p[1]$.

Example. For a scheme X, its étale $\mathbb{Z}_p$- and $\mathbb{Q}_p$-cohomology are $R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) := \mathrm{holim}\, R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n) = R\Gamma(X_{\text{ét}}, \mathbb{Z})\widehat{\otimes}\mathbb{Z}_p$, $R\Gamma_{\text{ét}}(X, \mathbb{Q}_p) := R\Gamma(X_{\text{ét}}, \mathbb{Z})\widehat{\otimes}\mathbb{Q}_p$.[4]

1.2. *The derived de Rham algebra.* For a morphism of commutative rings $A \to B$ we denote by $\Omega^\cdot_{B/A}$ the relative de Rham complex of B over A. This is a commutative dg A-algebra with components $\Omega^i_{B/A} = \Lambda^i_B \Omega_{B/A}$, where $\Omega_{B/A}$ is the B-module of relative Kähler differentials; it carries a ring filtration $F^n = \Omega^{\geq n}_{B/A}$.

[3] In loc. cit. E_∞ algebras are called "May algebras".

[4] Since $R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n) = R\Gamma(X_{\text{ét}}, \mathbb{Z}) \otimes^L \mathbb{Z}/p^n$.

We will use the F-completed version $L\Omega^{\cdot\,\hat{}}_{B/A}$ of Illusie's derived de Rham algebra defined in [Ill2, Ch. VIII, (2.1.3.3)]. To construct it, consider the canonical simplicial resolution $P. = P_A(B).$ of B from [Ill1, Ch. I,(1.5.5.6)]. This is a simplicial commutative A-algebra such that each P_i is a polynomial A-algebra. The de Rham complexes $\Omega^{\cdot}_{P./A}$ form a simplicial filtered commutative dg A-algebra, so the corresponding total complex $L\Omega^{\cdot}_{B/A}$ is a filtered commutative dg A-algebra (see [Ill1, Ch. I, 3.1.3]). Now $L\Omega^{\cdot\,\hat{}}_{B/A}$ is its completion with respect to the filtration $F^{\cdot}$. *Here "completion" is understood as a mere projective system of quotients modulo* F^i. One has a natural identification $\mathrm{gr}^i_F L\Omega^{\cdot\,\hat{}}_{B/A} \xrightarrow{\sim} (L\Lambda^i_B(\mathrm{L}_{B/A}))[-i]$ compatible with the product; here $\mathrm{L}_{B/A} := \Omega_{P./A} \otimes_{P.} B$ is the relative cotangent complex and $L\Lambda^i_B$ is the nonabelian left derived functor of the exterior power functor (see Ch. II and Ch. I of [Ill1]). For A-flat B's, the construction is compatible with base change. It is compatible with direct limits. If in the above definition we replace $P.$ by any simplicial A-algebra resolution of B whose terms are polynomial A-algebras, then the output is naturally quasi-isomorphic to $L\Omega^{\cdot\,\hat{}}_{B/A}$.

The next lemma is a particular case of [Ill1, Ch. I, 4.3.2.1(ii)]. For a flat B-module T we denote by $B\langle T\rangle^{\cdot}$ its divided powers symmetric algebra.

Lemma. *The complex $L\Lambda^i_B(T[1])$ is acyclic outside degree $-i$. There is a canonical isomorphism of graded B-algebras compatible with base change*

$$H^{-\cdot} L\Lambda^{\cdot}_B(T[1]) \xrightarrow{\sim} B\langle T\rangle^{\cdot}. \tag{1.2.1}$$

1.3. Let K be a p-adic field, i.e., a complete discretely-valued field of characteristic zero with perfect residue field k of characteristic $p > 0$, $\bar{K}$ be an algebraic closure of K, and O_K, $O_{\bar{K}}$ be rings of integers in K, $\bar{K}$. Let $K_0 \subset K$ be the field of fractions of the Witt vectors $W(k) = O_{K_0}$, and let $\mathfrak{a}$ be the fractional ideal in $\bar{K}$ generated by $p^{-\frac{1}{p-1}}\mathfrak{D}^{-1}_{K/K_0}$, where $\mathfrak{D}_{K/K_0}$ is the different. For an O_K-algebra B we often write $\Omega_B := \Omega_{B/O_K}$, $L\Omega^{\cdot\,\hat{}}_B := L\Omega^{\cdot\,\hat{}}_{B/O_K}$, $\mathrm{L}_B = \mathrm{L}_{B/O_K}$, etc.

The next key result is due to Fontaine [F2, Th 1]; we include a proof for completeness sake. Consider the map $\mu_{p^\infty} \subset O^\times_{\bar{K}} \xrightarrow{d\log} \Omega_{O_{\bar{K}}}$ and its $O_{\bar{K}}$-linear extension

$$(\bar{K}/O_{\bar{K}})(1) = O_{\bar{K}} \otimes \mu_{p^\infty} \to \Omega_{O_{\bar{K}}}. \tag{1.3.1}$$

Theorem. *One has $\mathrm{L}_{O_{\bar{K}}} \xrightarrow{\sim} \Omega_{O_{\bar{K}}}$, and (1.3.1) is surjective with kernel $(\mathfrak{a}/O_{\bar{K}})(1)$.*

Proof. If K'/K is a finite extension, then $O_{K'}/O_K$ is a complete intersection. So, if π is a generator of $O_{K'}/O_K$, $f(t)$ its minimal polynomial, then $\mathrm{L}_{O_{K'}}$ is the cone of multiplication by $f'(\pi)$ endomorphism of $O_{K'}$; hence $\mathrm{L}_{O_{K'}} \xrightarrow{\sim} \Omega_{O_{K'}}$. Passing to the limit, we get the first assertion. Let us prove the second one.

(i) By the above, $\Omega_{O_{K'}} \simeq O_{K'}/\mathfrak{D}_{K'/K}$. If K''/K' is another finite extension, then the standard exact triangle of the cotangent complexes reduces to a short exact sequence $0 \to O_{K''} \otimes_{O_{K'}} \Omega_{O_{K'}/O_K} \to \Omega_{O_{K''}/O_K} \to \Omega_{O_{K''}/O_{K'}} \to 0$.

(ii) Replacing K', K by $\bar{K}$, K_0 and passing to the limit, we get a short exact sequence $0 \to O_{\bar{K}} \otimes_{O_K} \Omega_{O_K/O_{K_0}} \to \Omega_{O_{\bar{K}}/O_{K_0}} \to \Omega_{O_{\bar{K}}/O_K} \to 0$. Thus it suffices to prove the theorem for $K = K_0$, which we now assume.

(iii) Set $T := \mathrm{Ker}((\bar{K}/O_{\bar{K}})(1) \to \Omega)$, $F := K(\mu_p)$. The set of $O_{\bar{K}}$-submodules of $(\bar{K}/O_{\bar{K}})(1)$ is totally ordered by inclusion. Thus, since $O_{\bar{K}} \otimes_{O_F} \Omega_{O_F} \subset \Omega_{O_{\bar{K}}}$ is a nonzero $O_{\bar{K}}$-module generated by $d\log(\mu_p)$, one has $T \subset (p^{-1}O_{\bar{K}}/O_{\bar{K}})(1) = O_{\bar{K}} \otimes \mu_p$. Since Ω_{O_F} is isomorphic to $O_F/p^{1-\frac{1}{p-1}}O_F$, one has $T = (p^{-\frac{1}{p-1}}O_{\bar{K}}/O_{\bar{K}})(1)$.

(iv) It remains to prove surjectivity of $(\bar{K}/O_{\bar{K}})(1) \to \Omega_{O_{\bar{K}}}$. Let $K' \subset \bar{K}$ be any finite extension of K; we want to check that $\Omega_{O_{K'}} \subset \Omega_{O_{\bar{K}}}$ lies in $O_{\bar{K}} d\log(\mu_{p^\infty})$. Suppose p^n kills $\Omega_{O_{K'}}$. Let us show that $\Omega_{O_{K'}} \subset O_{\bar{K}} d\log(\mu_{p^{n+1}})$. Set $K'' := K'(\mu_{p^{n+1}})$. The set of $O_{K''}$-submodules of $\Omega_{O_{K''}}$ is totally ordered. Thus, since $p^n d\log(\mu_{p^{n+1}}) \neq 0$ by (iii), $\Omega_{O_{K'}}$ lies in $O_{K''} d\log(\mu_{p^{n+1}})$, q.e.d. □

1.4. *For a complex P acyclic in degrees $\neq 0$, we often write P instead of $H^0 P$.*

Consider the filtered commutative dg O_K-algebra $\mathrm{A}_{\mathrm{dR}} = \mathrm{A}_{\mathrm{dR}\,\bar{K}/K} := L\Omega^{\cdot}_{O_{\bar{K}}/O_K}$ and the corresponding filtered E_∞ O_K-algebra $\mathrm{A}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p$ (see §1.1). Let us describe the graded $O_{\bar{K}}$-algebras $\mathrm{gr}^{\cdot}_F \mathrm{A}_{\mathrm{dR}}$, $\mathrm{gr}^{\cdot}_F \mathrm{A}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p$.

Proposition. *(i) The complexes $gr^i_F A_{dR} \widehat{\otimes} \mathbb{Z}_p$ are acyclic in nonzero degrees, and there is a canonical isomorphism of graded algebras*

$$gr^{\cdot}_F A_{dR} \widehat{\otimes} \mathbb{Z}_p \xrightarrow{\sim} \hat{O}_{\bar{K}} \langle \hat{\mathfrak{a}}(1) \rangle^{\cdot}. \tag{1.4.1}$$

(ii) One has $gr^0_F A_{dR} = A_{dR}/F^1 = O_{\bar{K}}$, and the complexes $gr^i_F A_{dR}$ for $i > 0$ are acyclic in degrees $\neq 1$. There are natural isomorphisms of $O_{\bar{K}}$-modules

$$\Omega^{\langle i \rangle} := H^1 gr^i_F A_{dR} \xrightarrow{\sim} (\bar{K}/i!^{-1}\mathfrak{a}^i)(i) = (\mathbb{Q}_p/\mathbb{Z}_p) \otimes i!^{-1} \hat{\mathfrak{a}}^i(i). \tag{1.4.2}$$

Proof. (i) By the theorem in §1.3, one has $\mathrm{L}_{O_{\bar{K}}/O_K} \xrightarrow{\sim} \Omega_{O_{\bar{K}}} \xrightarrow{\sim} (\bar{K}/\mathfrak{a})(1) = (\mathbb{Q}_p/\mathbb{Z}_p) \otimes \mathfrak{a}(1)$; hence $\mathrm{gr}^i_F \mathrm{A}_{\mathrm{dR}} \xrightarrow{\sim} L\Lambda^i_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p)[-i] \otimes \mathfrak{a}^i(i)$. One has $L\Lambda^i_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p) \otimes^L (\mathbb{Z}/p^n) = L\Lambda^i_{\mathbb{Z}/p^n}((\mathbb{Q}_p/\mathbb{Z}_p) \otimes^L (\mathbb{Z}/p^n)) = L\Lambda^i_{\mathbb{Z}/p^n}((\mathbb{Z}/p^n)[1])$, which identifies with $i!^{-1}(\mathbb{Z}/p^n)[i]$ in a way compatible with the product by (1.2.1). Therefore $\mathrm{gr}^{\cdot}_F \mathrm{A}_{\mathrm{dR}} \otimes^L (\mathbb{Z}/p^n) \xrightarrow{\sim} \mathbb{Z}/p^n \langle (\mathfrak{a}/p^n \mathfrak{a})(1) \rangle^{\cdot}$, which yields (1.4.1).

(ii) follows from (i) by the next observation (applied to $C = \mathrm{gr}^i_F \mathrm{A}_{\mathrm{dR}}$, with (1.4.2) defined by the condition that $T_p(1.4.2) = (1.4.1)$): If a complex C of abelian groups has p-torsion cohomology and $H^{\neq 0}(C \otimes^L \mathbb{Z}/p) = 0$, then $H^1 C$ is p-divisible and $H^{\neq 1} C = 0$.[5] □

1.5. By 1.4(i), the algebras $(\mathrm{A}_{\mathrm{dR}}/F^i) \widehat{\otimes} \mathbb{Z}_p$, hence $(\mathrm{A}_{\mathrm{dR}}/F^i) \widehat{\otimes} \mathbb{Q}_p$, are acyclic in nonzero degree. By loc. cit., $(\mathrm{A}_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Q}_p$ is an i-truncated dvr with residue field $\mathbb{C}_p := \hat{O}_{\bar{K}} \otimes \mathbb{Q}$, so $\mathrm{A}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p := \varprojlim (\mathrm{A}_{\mathrm{dR}}/F^i) \widehat{\otimes} \mathbb{Q}_p$ is a dvr. Let $\mathfrak{m}_{\mathrm{dR}}$ be its maximal ideal; (1.4.1) yields a canonical identification $\mathfrak{m}_{\mathrm{dR}}/\mathfrak{m}^2_{\mathrm{dR}} = \mathrm{gr}^1_F \mathrm{A}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p \xrightarrow{\sim} \mathbb{C}_p(1)$.

Proposition. *There is a canonical ring isomorphism of filtered rings*

$$u_{\mathbb{Q}} : B^+_{dR} \xrightarrow{\sim} A_{dR} \widehat{\otimes} \mathbb{Q}_p. \tag{1.5.1}$$

Proof. The ring $(\mathrm{A}_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Z}_p$ is an infinitesimal p-adic O_K-thickening of $\hat{O}_K = (\mathrm{A}_{\mathrm{dR}}/F^1) \widehat{\otimes} \mathbb{Z}_p$ of order $\leq i$ (see [F3, 1.1]). Let $\mathrm{A}_{\mathrm{inf}}/F^{i+1}$ be the universal thickening ([F3, 1.3]); we have a canonical map $u_i : \mathrm{A}_{\mathrm{inf}}/F^{i+1} \to (\mathrm{A}_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Z}_p$. Since $\mathrm{B}^+_{\mathrm{dR}}/F^{i+1} := (\mathrm{A}_{\mathrm{inf}}/F^{i+1}) \otimes \mathbb{Q}$ is an i-truncated dvr and u_1 is an isomorphism by [F3, 1.4.3], $u_{i\mathbb{Q}} : \mathrm{B}^+_{\mathrm{dR}}/F^{i+1} \xrightarrow{\sim} (\mathrm{A}_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Q}_p$. Set $u_{\mathbb{Q}} := \varprojlim u_{i\mathbb{Q}}$. □

Remarks. (i) The map $\mathrm{A}_{\mathrm{dR}} \to \mathrm{A}_{\mathrm{dR}}/F^1 = O_{\bar{K}}$ yields an isomorphism $\mathrm{A}_{\mathrm{dR}} \otimes \mathbb{Q} \xrightarrow{\sim} \bar{K}$. Thus the morphism $\mathrm{A}_{\mathrm{dR}} \otimes \mathbb{Q} \to \mathrm{A}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p$ equals the usual embedding $\bar{K} \hookrightarrow \mathrm{B}^+_{\mathrm{dR}}$.

[5] Use the fact that every complex of abelian groups splits, i.e., is quasi-isomorphic to a complex with zero differential.

(ii) For a finite extension K'/K, $K'\subset\bar K$, the evident map $A_{\mathrm{dR}\,\bar K/K}\to A_{\mathrm{dR}\,\bar K/K'}$ yields an isomorphism $A_{\mathrm{dR}\,\bar K/K}\widehat\otimes\mathbb{Q}_p\xrightarrow{\sim}A_{\mathrm{dR}\,\bar K/K'}\widehat\otimes\mathbb{Q}_p$ compatible with (1.5.1).

1.6**.** The next result, which will not be used in the rest of the article, is a reinterpretation of Colmez's theorem [Col]. It would be nice to find a simpler direct proof.

Proposition. *The complexes A_{dR}/F^i are acyclic in nonzero degrees; the maps $H^0(A_{dR}/F^{i+1})\to H^0(A_{dR}/F^i)$ are injective. Set $O^{\langle i\rangle}:=H^0(A_{dR}/F^{i+1})$; thus $O_{\bar K}=O^{\langle 0\rangle}\supset O^{\langle 1\rangle}\supset\dots$ and $(A_{dR}/F^{i+1})\widehat\otimes\mathbb{Z}_p$ is equal to the p-adic completion $\hat O^{\langle i\rangle}$ of $O^{\langle i\rangle}$.*

Proof. By 1.4(ii), the exact cohomology sequence for $0\to\mathrm{gr}^i_F\mathrm{A}_{\mathrm{dR}}\to\mathrm{A}_{\mathrm{dR}}/F^{i+1}\to\mathrm{A}_{\mathrm{dR}}/F^i\to 0$ reduces to $0\to O^{\langle i\rangle}\to O^{\langle i-1\rangle}\xrightarrow{d^{\langle i\rangle}}\Omega^{\langle i\rangle}\to H^1(\mathrm{A}_{\mathrm{dR}}/F^{i+1})\to H^1(\mathrm{A}_{\mathrm{dR}}/F^i)\to 0$. So $O_{\bar K}=O^{\langle 0\rangle}\supset O^{\langle 1\rangle}\supset\dots$, and the vanishing of $H^1(\mathrm{A}_{\mathrm{dR}}/F^{i+1})$ amounts to that of $H^1(\mathrm{A}_{\mathrm{dR}}/F^i)$ combined with surjectivity of $d^{\langle i\rangle}:O^{\langle i-1\rangle}\to\Omega^{\langle i\rangle}$. It remains to prove that all $d^{\langle i\rangle}$ are surjective.

Recall that Colmez [Col] considers a sequence of subalgebras $O_{\bar K}=O^{(0)}\supset O^{(1)}\supset\dots$ and derivations $d^{(i)}:O^{(i-1)}\to\Omega^{(i)}$ defined by induction: $d^{(i)}$ is a universal O_K-linear derivation with values in an $O_{\bar K}$-module, and $O^{(i)}:=\mathrm{Ker}\,d^{(i)}$. An induction by i shows that $O^{\langle i\rangle}\supset O^{(i)}$: Indeed, $\Omega^{\langle i\rangle}$ are $O_{\bar K}$-modules and $d^{\langle i\rangle}:O^{\langle i-1\rangle}\to\Omega^{\langle i\rangle}$ is a derivation; so, if $O^{\langle i-1\rangle}\supset O^{(i-1)}$, then $d^{\langle i\rangle}|_{O^{(i-1)}}=a^{(i)}d^{(i)}$ for some $O_{\bar K}$-linear map $a^{(i)}:\Omega^{(i)}\to\Omega^{\langle i\rangle}$; thus $O^{\langle i\rangle}\supset O^{(i)}$.

Let i be the smallest number such that $d^{\langle i\rangle}$ is not surjective. Since $E:=\Omega^{\langle i\rangle}/d^{\langle i\rangle}(O^{\langle i-1\rangle})$ is p-torsion p-divisible, one has $E\widehat\otimes\mathbb{Q}_p=T_pE\otimes\mathbb{Q}\neq 0$. Applying $\cdot\widehat\otimes\mathbb{Z}_p$ to the exact triangle $O^{\langle i\rangle}\to\mathrm{A}_{\mathrm{dR}}/F^{i+1}\to E[-1]$, we get a short exact sequence $0\to\hat O^{\langle i\rangle}\to(\mathrm{A}_{\mathrm{dR}}/F^{i+1})\widehat\otimes\mathbb{Z}_p\to T_pE\to 0$. By [Col], $\mathrm{A}_{\mathrm{inf}}/F^{i+1}=\hat O^{(i)}$. By universality, the map $u_i:\mathrm{A}_{\mathrm{inf}}/F^{i+1}\to(\mathrm{A}_{\mathrm{dR}}/F^{i+1})\widehat\otimes\mathbb{Z}_p$ equals the composition $\hat O^{(i)}\to\hat O^{\langle i\rangle}\hookrightarrow(\mathrm{A}_{\mathrm{dR}}/F^{i+1})\widehat\otimes\mathbb{Z}_p$, so its composition with the projection onto T_pE vanishes. This cannot happen since $u_{i\mathbb{Q}}$ is an isomorphism (see §1.5), q.e.d. □

2. h-topology and semi-stable compactifications

2.1**.** *A topological digression.* The next proposition is a generalization of [V2, 4.1].

Let $\mathcal{V}$ be an essentially small site. As in [V1], we denote by $\mathcal{V}^\sim$ the corresponding topos (the category of sheaves of sets on $\mathcal{V}$).

For us, a *base for* $\mathcal{V}$ is a pair $(\mathcal{B},\phi)$, where $\mathcal{B}$ is an essentially small category and $\phi:\mathcal{B}\to\mathcal{V}$ is a *faithful* functor, that satisfies the next property:
$(*)$ *For any $V\in\mathcal{V}$ and a finite family of pairs (B_α,f_α), $B_\alpha\in\mathcal{B}$, $f_\alpha:V\to\phi(B_\alpha)$, there exists a set of objects $B'_\beta\in\mathcal{B}$ and a covering family $\{\phi(B'_\beta)\to V\}$ such that every composition $\phi(B'_\beta)\to V\to\phi(B_\alpha)$ lies in $Hom(B'_\beta,B_\alpha)\subset Hom(\phi(B'_\beta),\phi(B_\alpha))$.*

Remarks. (i) Property $(*)$ for an empty set of (B_α,f_α)'s means that every $V\in\mathcal{V}$ has a covering by objects $\phi(B)$, $B\in\mathcal{B}$. If ϕ is fully faithful, then $(*)$ amounts to this assertion.[6]
(ii) If $\mathcal{B}$ admits finite products and ϕ commutes with finite products, then it suffices to check $(*)$ for families (B_α,f_α) having ≤ 1 element.

[6] The proposition below in this situation amounts to [V2, 4.1].

(iii) In the general case, it suffices to check $(*)$ for families (B_α, f_α) having ≤ 2 elements.

Suppose $(\mathcal{B}, \phi)$ is a base for $\mathcal{V}$. Define a covering sieve in $\mathcal{B}$ as a sieve whose ϕ-image is a covering family in $\mathcal{V}$.

Proposition. *(i) Covering sieves in $\mathcal{B}$ form a Grothendieck topology on $\mathcal{B}$.*
(ii) The functor $\phi : \mathcal{B} \to \mathcal{V}$ is continuous (see [V2, 1.1]*).*
(iii) ϕ yields an equivalence of the toposes: one has $\mathcal{B}^\sim \xrightarrow{\sim} \mathcal{V}^\sim$.

We call the above topology on $\mathcal{B}$ the *ϕ-induced* topology.[7]

Proof. (i) Let us check that covering sieves in $\mathcal{B}$ are stable with respect to pullback; the rest of the axioms from [V1, 1.1] are evident. For a morphism $g : B' \to B$ in $\mathcal{B}$ and a covering sieve $\mathfrak{s}$ on B, let us find a covering family on B' that belongs to the g-pullback of $\mathfrak{s}$. The $\phi(g)$-pullback of $\phi(\mathfrak{s})$ is a covering sieve in $\mathcal{V}$, so there is a covering family $\{\pi_\gamma : V_\gamma \to \phi(B')\}$ such that every composition $V_\gamma \to \phi(B') \to \phi(B)$ can be factored as $V_\gamma \xrightarrow{g_\gamma} \phi(B_\gamma) \xrightarrow{\phi(p_\gamma)} \phi(B)$, where $p_\gamma : B_\gamma \to B$ belong to $\mathfrak{s}$. Applying $(*)$ to V_γ and (B', π_γ), (B_γ, g_γ), we find a covering family $\{\phi(B'_{\beta_\gamma}) \to V_\gamma\}$ as in $(*)$. The composite covering $\{\phi(B'_{\beta_\gamma}) \to \phi(B')\}$ comes then from a covering family $\{B'_{\beta_\gamma} \to B'\}$ in $\mathcal{B}$ which lies in the g-pullback of $\mathfrak{s}$.

(ii) We know that ϕ sends covering families to covering families, so it suffices to show that for any given $p_\alpha : B_\alpha \to B$ in $\mathcal{B}$ and $f_\alpha : V \to \phi(B_\alpha)$, $\alpha = 1, 2$, such that $\phi(p_1)f_1 = \phi(p_2)f_2$ there is a covering $\{\pi_\beta : V_\beta \to V\}$ and morphisms $\xi_{\alpha\beta} : B'_\beta \to B_\alpha$, $g_\beta : V_\beta \to \phi(B'_\beta)$ such that $p_1\xi_{1\beta} = p_2\xi_{2\beta}$ and $\phi(\xi_{\alpha\beta})g_\beta = f_\alpha\pi_\beta$. Such a datum (with g_β the identity map) comes from $(*)$ applied to V and $(B_1, f_1), (B_2, f_2)$.

(iii) By (ii), one has the usual adjoint functors between the categories of sheaves $(\phi^s, \phi_s) : \mathcal{B}^\sim \leftrightarrows \mathcal{V}^\sim$. To prove that they are mutually inverse equivalences, we will check that for $\mathcal{F} \in \mathcal{B}^\sim$ and $\mathcal{G} \in \mathcal{V}^\sim$ the adjunction maps $a_\mathcal{F} : \mathcal{F} \to \phi_s\phi^s\mathcal{F}$, $b_\mathcal{G} : \phi^s\phi_s\mathcal{G} \to \mathcal{G}$ are isomorphisms.

Recall that $\phi^s\mathcal{F} = (\phi^\cdot\mathcal{F})^\sim$, where $\phi^\cdot$ is the pullback of presheaves and $^\sim$ is the sheafification functor. For $V \in \mathcal{V}$ one has $(\phi^\cdot\mathcal{F})(V) = \operatorname{colim}_{\mathcal{C}(V)}\mathcal{F}$, where $\mathcal{C}(V)$ is the category of pairs (B, f), $B \in \mathcal{B}$, $f : V \to \phi(B)$, with $\operatorname{Hom}_{\mathcal{C}(V)}((B, f), (B', f')) := \{g \in \operatorname{Hom}(B', B) : \phi(g)f' = f\}$, and we set $\mathcal{F}(B, f) := \mathcal{F}(B)$.

(a) To show that $a_\mathcal{F}$ is an isomorphism, we check that it is injective and surjective:

$a_\mathcal{F}$ is injective: Suppose we have $B \in \mathcal{B}$ and $\xi_1, \xi_2 \in \mathcal{F}(B)$ such that $a_\mathcal{F}(\xi_1) = a_\mathcal{F}(\xi_2)$; let us show that the ξ_i coincide. One has

$$a_\mathcal{F}(\xi_i) \in (\phi_s\phi^s\mathcal{F})(B) = (\phi^s\mathcal{F})(\phi(B)),$$

and the equality means that there is a covering $\{\pi_\gamma : V_\gamma \to \phi(B)\}$ such that the images of ξ_i in $(\phi^\cdot\mathcal{F})(V_\gamma) = \operatorname{colim}_{\mathcal{C}(V_\gamma)}\mathcal{F}$ coincide. Thus for some *finite* subdiagram $\mathcal{C}(V_\gamma)' \subset \mathcal{C}(V_\gamma)$ that contains (B, π_γ) the images of ξ_i in $\operatorname{colim}_{\mathcal{C}(V_\gamma)'}\mathcal{F}$ coincide. Applying $(*)$ to V_γ and pairs from $\mathcal{C}(V_\gamma)'$, we get a covering $\{\phi(B'_{\beta_\gamma}) \to V_\gamma\}$ such that the image of $\mathcal{C}(V_\gamma)'$ in each $\mathcal{C}(\phi(B'_{\beta_\gamma}))$ comes from a diagram in $\mathcal{B}^\circ/B'_{\beta_\gamma}$. The composite covering $\{\phi(B'_{\beta_\gamma}) \to \phi(B)\}$ comes then from a covering $\{B'_{\beta_\gamma} \to B\}$ in $\mathcal{B}$, and the images of ξ_i in $\mathcal{F}(B'_{\beta_\gamma})$ coincide. Then $\xi_1 = \xi_2$ since $\mathcal{F}$ is a sheaf, q.e.d.

[7]The terminology is compatible with that of [V2, 3.1].

$a_{\mathcal{F}}$ *is surjective*: For $B \in \mathcal{B}$, $\chi \in (\phi_s\phi^s\mathcal{F})(B)$ we look for a covering $\{B'_\beta \to B\}$ in $\mathcal{B}$ such that $\chi|_{B'_\beta}$ lies in the image of $\mathcal{F}(B'_\beta) \to (\phi_s\phi^s\mathcal{F})(B'_\beta)$. To find it, consider χ as an element of $(\phi^s\mathcal{F})(\phi(B))$. There is a covering $\{\pi_\gamma : V_\gamma \to \phi(B)\}$ such that $\chi|_{V_\gamma}$ lies in the image of $(\phi^{\cdot}\mathcal{F})(V_\gamma) \to (\phi^s\mathcal{F})(V_\gamma)$; i.e., one has $f_\gamma : V_\gamma \to \phi(B_\gamma)$ such that $\chi|_{V_\gamma}$ lies in the image of the composition $\mathcal{F}(B_\gamma) \to (\phi^s\mathcal{F})(\phi(B_\gamma)) \to (\phi^s\mathcal{F})(V_\gamma)$, where the second arrow comes from f_γ. Applying $(*)$ to V_γ and (B, π_γ), (B_γ, f_γ), we find a covering $\{\phi(B'_{\beta_\gamma}) \to V_\gamma\}$ as in $(*)$; the composite covering $\{\phi(B'_{\beta_\gamma}) \to \phi(B)\}$ comes then from a covering $\{B'_{\beta_\gamma} \to B\}$ that satisfies the promised property.

(b) $b_{\mathcal{G}}$ *is an isomorphism*: Since $\phi_s(b_{\mathcal{G}})a_{\phi_s\mathcal{G}} = \mathrm{id}_{\phi_s\mathcal{G}}$ and we already know that $a_{\phi_s\mathcal{G}}$ is an isomorphism, we see that $\phi_s(b_{\mathcal{G}}) : \phi_s\phi^s\phi_s(\mathcal{G}) \to \phi_s\mathcal{G}$ is an isomorphism. Thus $b_{\mathcal{G}}(B) : \phi^s\phi_s\mathcal{G}(\phi(B)) \to \mathcal{G}(\phi(B))$ is an isomorphism for every $B \in \mathcal{B}$. Since every $V \in \mathcal{V}$ admits a covering by objects $\phi(B)$, $B \in \mathcal{B}$, this implies that $b_{\mathcal{G}}$ is both injective and surjective, hence an isomorphism, q.e.d. $\square$

Exercises. (i) For any presheaf $\mathcal{J}$ on $\mathcal{V}$ one has $\phi_s(\mathcal{J}^{\sim}) = (\phi_\cdot\mathcal{J})^{\sim}$.
(ii) Suppose $(\mathcal{B}, \phi)$ is a base for $\mathcal{V}$ and $(\mathcal{B}', \phi')$ is a base for the ϕ-induced topology on $\mathcal{B}$. Then $(\mathcal{B}', \phi\phi')$ is a base for $\mathcal{V}$.

2.2. For a field K, let $\mathcal{V}ar_K$ be the category of K-varieties, i.e., reduced separated K-schemes of finite type. We will consider categories $\mathcal{B}$ formed by varieties equipped with appropriate compactifications, referred to as *pairs*:

(a) *Geometric setting:* Let $j : U \hookrightarrow \bar{U}$ be an open embedding such that $\bar{U}$ is proper and U is dense in $\bar{U}$. We call such a datum a *geometric pair over* K, or *geometric* K*-pair*, and denote it by $(U, \bar{U})$. We say that $(U, \bar{U})$ is a regular normal crossings pair, *nc-pair* for short, if $\bar{U}$ is a regular scheme and $\bar{U} \smallsetminus U$ is a divisor with normal crossings in $\bar{U}$; it is a *strict* nc-pair if the irreducible components of $\bar{U} \smallsetminus U$ are regular. A morphism $f : (U', \bar{U}') \to (U, \bar{U})$ of pairs is a map $\bar{U}' \to \bar{U}$ which sends U' to U. We denote the category of geometric K-pairs by $\mathcal{V}ar_K^{\mathrm{c}}$; let $\mathcal{V}ar_K^{\mathrm{nc}}$ be the full subcategory of nc-pairs.

(b) *Arithmetic* K*-setting:* Suppose K is a p-adic field as in §1.3. An *arithmetic pair over* K, a.k.a. *arithmetic* K*-pair*, is an open embedding $j : U \hookrightarrow \bar{U}$ with dense image of a K-variety U into a reduced proper flat O_K-scheme $\bar{U}$.

For such a $(U, \bar{U})$ we set $O_{K_U} := \Gamma(\bar{U}, \mathcal{O}_{\bar{U}})$, $K_U := \Gamma(\bar{U}_K, \mathcal{O}_{\bar{U}})$. Then K_U is the product of several finite extensions of K (labeled by the connected components of $\bar{U}_K$; if $\bar{U}$ is normal, then O_{K_U} is the product of the corresponding rings of integers. *The closed fiber* $\bar{U}_s$ of $\bar{U}$ is the union of fibers over the closed points of O_{K_U}.

We say that $(U, \bar{U})$ is a *semi-stable pair*, or simply *ss-pair*, if (i) $\bar{U}$ is a regular scheme, (ii) $\bar{U} \smallsetminus U$ is a divisor with normal crossings on $\bar{U}$, and (iii) the closed fiber $\bar{U}_s$ is reduced. Our ss-pair is *strict* if the irreducible components of $\bar{U} \smallsetminus U$ are regular. Arithmetic K-pairs form a category $\mathcal{V}ar_K^{\mathrm{cc}}$; let $\mathcal{V}ar_K^{\mathrm{ss}}$ be the full subcategory of ss-pairs.

(c) *Arithmetic* $\bar{K}$*-setting:* For K as in (b), let $\bar{K}$ be its algebraic closure. An *arithmetic pair over* $\bar{K}$, a.k.a. *arithmetic* $\bar{K}$*-pair*, is an open embedding $j : V \hookrightarrow \bar{V}$ with dense image of a $\bar{K}$-variety V into a reduced proper flat $O_{\bar{K}}$-scheme $\bar{V}$. A connected $(V, \bar{V})$ is said to be *semi-stable*, a.k.a. *ss-pair*, if there exists an ss-pair $(U, \bar{U})$ over K and a $\bar{K}$-point $\alpha : K_U \to \bar{K}$ (see (b)) such that $(V, \bar{V})$ is isomorphic to $(U, \bar{U})_\alpha = (U_{\bar{K}}, \bar{U}_{O_{\bar{K}}}) := (U \otimes_{K_U} \bar{K}, \bar{U} \otimes_{O_{K_U}} O_{\bar{K}})$. Then $\bar{V}$ is normal (say, by Serre's criterion). An arbitrary $(V, \bar{V})$ is semi-stable if such are all its connected components. Denote by $\mathcal{V}ar^{\mathrm{cc}}_{\bar{K}}$ the category of all arithmetic pairs over $\bar{K}$, and by $\mathcal{V}ar^{\mathrm{ss}}_{\bar{K}} = \mathcal{V}ar^{\mathrm{ss}}_{\bar{K}/K}$ its full subcategory of ss-pairs.

Remark. If K' is a finite extension of K contained in $\bar{K}$, then $\mathcal{V}ar^{\mathrm{ss}}_{\bar{K}/K'} \subset \mathcal{V}ar^{\mathrm{ss}}_{\bar{K}/K}$. For all the constructions below the difference between them is irrelevant.

These categories are connected by commutative diagrams of functors

$$\begin{array}{ccccc} \mathcal{V}ar^{\mathrm{cc}}_K & \to & \mathcal{V}ar^{\mathrm{c}}_K & \to & \mathcal{V}ar_K, \\ \uparrow & & \uparrow & & \\ \mathcal{V}ar^{\mathrm{ss}}_K & \to & \mathcal{V}ar^{\mathrm{nc}}_K & & \end{array} \qquad \begin{array}{ccccc} \mathcal{V}ar^{\mathrm{cc}}_{\bar{K}} & \to & \mathcal{V}ar^{\mathrm{c}}_{\bar{K}} & \to & \mathcal{V}ar_{\bar{K}}, \\ \uparrow & & \uparrow & & \\ \mathcal{V}ar^{\mathrm{ss}}_{\bar{K}} & \to & \mathcal{V}ar^{\mathrm{nc}}_{\bar{K}} & & \end{array} \tag{2.2.1}$$

where the vertical arrows are the fully faithful embeddings, and the upper horizontal lines are faithful forgetful functors of passing to the generic fiber and $(U, \bar{U}) \mapsto U$. The K- and $\bar{K}$-settings are connected by base change functors

$$\begin{array}{ccccc} \mathcal{V}ar^{\mathrm{ss}}_{\bar{K}} & \to & \mathcal{V}ar^{\mathrm{nc}}_{\bar{K}} & \to & \mathcal{V}ar_{\bar{K}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{V}ar^{\mathrm{ss}}_K & \to & \mathcal{V}ar^{\mathrm{nc}}_K & \to & \mathcal{V}ar_K. \end{array} \tag{2.2.2}$$

Here the two right vertical arrows are the evident base change $\cdot \otimes_K \bar{K}$, and the left one assigns to a semi-stable K-pair $(U, \bar{U})$ the disjoint sum of pairs $(U, \bar{U})_\alpha$ for all $\bar{K}$-points $\alpha : K_U \to \bar{K}$.

2.3. A morphism $f : (V, \bar{V}) \to (U, \bar{U})$ of pairs in either of the settings of §2.2 is called an *alteration* (of $(U, \bar{U})$) if $f^{-1}(U) = V$, the generic fibers of f are zero-dimensional, and their union is dense in V. In setting (a), f is a *(strict) nc-alteration* if $(V, \bar{V})$ is a (strict) nc-pair; in settings (b), (c), f is a *(strict) ss-alteration* if $(V, \bar{V})$ is a (strict) ss-pair.

If f is an alteration, then $f|_V : V \to U$ is proper and surjective; the composition of alterations is an alteration.

Here is a key result of de Jong [dJ1, 4.1, 6.5]:

Theorem. *Every geometric pair admits a strict nc-alteration. Every arithmetic pair, either over K or over $\bar{K}$, admits a strict ss-alteration. The alterations can be chosen so that $\bar{V}$ is projective.*

Remark. Our conventions slightly differ from de Jong's: he understands varieties to be irreducible and semi-stable K-pairs $(U, \bar{U})$ to have property $K_U = K$; his notation for $(U, \bar{U})$ is $(\bar{U}, Z)$, $Z := \bar{U} \smallsetminus U$.

2.4. For a field K, the *h-topology* (see [SV]) on $\mathcal{V}ar_K$ is generated by the pretopology whose coverings are finite families of maps $\{Y_\alpha \to X\}$ such that $Y := \amalg Y_\alpha \to X$ is a universal topological epimorphism.[8] It is stronger than the étale and proper topologies.[9] We denote the h-site by $\mathcal{V}ar_{K\mathrm{h}}$; the h-site of X is denoted by X_{h}.

Exercise. Let $f : Y \to X$ be a morphism in $\mathcal{V}ar_K$.
(i) f is an h-covering if (and only if) for every irreducible curve $C \subset X$ the base change $Y_{\tilde{C}} \to \tilde{C}$, where $\tilde{C}$ is the normalization of C, is an h-covering.[10]

[8] This means that a subset of X is Zariski open if (and only if) its preimage in Y is open, and the same is true after any base change.

[9] The latter is generated by a pretopology whose coverings are proper surjective maps.

[10] Hint: For an open $V \subset Y$ its image in X is constructible (EGA IV 1.8.4), so to show that $f(V)$ is open it suffices to check that for any curve $C \subset X$ the intersection $C \cap f(V)$ is open in C.

(ii) If X is a regular curve, then f is an h-covering if (and only if) the closure of the generic fiber of f maps onto X, or, equivalently, f is a covering for the flat topology.

Remark. By [SV, 10.4], every h-covering is a Zariski covering locally in proper topology. Therefore (see [D], [SD], or [Con]) h-coverings are morphisms of universal cohomological descent for torsion étale sheaves; if $K = \mathbb{C}$, then h-coverings are morphisms of universal cohomological descent for arbitrary sheaves on the classical topology. In particular, for any h-hypercovering $Y.$ of X and an abelian group A the canonical map $R\Gamma(X_{\text{ét}}, A) \to R\Gamma(Y_{\cdot\,\text{ét}}, A)$ (:= the total complex of the cosimplicial system of complexes $R\Gamma(Y_{i\,\text{ét}}, A)$) is a quasi-isomorphism if A is a torsion group. Passing to the limit, we see that $R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \xrightarrow{\sim} R\Gamma_{\text{ét}}(Y., \mathbb{Z}_p)$. If $K = \mathbb{C}$, then $R\Gamma(X_{\text{cl}}, A) \xrightarrow{\sim} R\Gamma(Y_{\cdot\text{cl}}, A)$ for any A. Since h-topology is stronger than the étale one, we see that $R\Gamma(X_{\text{ét}}, A) \xrightarrow{\sim} R\Gamma(X_{\text{h}}, A)$ if A is a torsion group (see [SV, 10.7] for a direct proof).

2.5. Let ϕ be the forgetful functor $(U, \bar{U}) \mapsto U$ on any of the categories $\mathcal{V}ar^?$ in §2.2.

Proposition. *If $\mathcal{V}ar^?$ is either of the categories $\mathcal{V}ar^c_K$, $\mathcal{V}ar^{nc}_K$, $\mathcal{V}ar^{cc}_K$, $\mathcal{V}ar^{\text{ss}}_K$, then $(\mathcal{V}ar^?, \phi)$ is a base for $\mathcal{V}ar_{Kh}$. If $\mathcal{V}ar^?$ is $\mathcal{V}ar^{cc}_{\bar{K}}$ or $\mathcal{V}ar^{\text{ss}}_{\bar{K}}$, then $(\mathcal{V}ar^?, \phi)$ is a base for $\mathcal{V}ar_{\bar{K}h}$.*

Proof. We consider the arithmetic K-setting, leaving the other two settings for the reader. Let us show that $(\mathcal{V}ar^{\text{cc}}_K, \phi)$ satisfies condition $(*)$ from §2.1. Our datum is a K-variety V and a finite collection of arithmetic K-pairs $(U_\alpha, \bar{U}_\alpha)$ and maps $f_\alpha : V \to U_\alpha$. We need to find an h-covering $\pi : V' \to V$ and an arithmetic pair $(V', \bar{V}')$ such that $f_\alpha \pi$ extend to maps $(V', \bar{V}') \to (U_\alpha, \bar{U}_\alpha)$. First we find an h-covering $V' \to V$ such that V' sits in some arithmetic K-pair $(V', \tilde{V}')$: let V'/V be a quasi-projective modification of V provided by the Chow lemma, and take for $\tilde{V}'$ the closure of V' in a projective space.[11] Then take $\bar{V}'$ to be the closure of the image of V' by the embedding $V' \hookrightarrow \tilde{V}' \times \Pi \bar{U}_\alpha$, and we are done.

To show that $(\mathcal{V}ar^{\text{ss}}_K, \phi)$ is a base for $\mathcal{V}ar_{K\text{h}}$, it suffices to check that $(\mathcal{V}ar^{\text{ss}}_K, \iota)$, where ι is the embedding $\mathcal{V}ar^{\text{ss}}_K \hookrightarrow \mathcal{V}ar^{\text{cc}}_K$, is a base for the ϕ-induced topology on $\mathcal{V}ar^{\text{cc}}_K$ (see Exercise (ii) in §2.1). Since ι is fully faithful, it suffices to check that for every $(U, \bar{U}) \in \mathcal{V}ar^{\text{cc}}_K$ there exists a map $(U', \bar{U}') \to (U, \bar{U})$ such that $U' \to U$ is an h-covering and $(U', \bar{U}')$ is semi-stable. Such a datum is provided by the de Jong theorem in §2.3, and we are done. $\square$

We call the ϕ-induced topology on either of the categories $\mathcal{V}ar^?$ the *h-topology*.

Remarks. (i) Any h-covering of $(U, \bar{U}) \in \mathcal{V}ar^{\text{ss}}_K$ has a refinement with terms of the same dimension as U (indeed, the same assertion in $\mathcal{V}ar_K$ is true by [SV, 10.4]; to pass to $\mathcal{V}ar^{\text{ss}}_K$, we apply the constructions from the proof above, and they preserve the dimension).
(ii) The proposition remains true if we replace the category of ss- or nc-pairs by its subcategory of strict pairs $(U, \bar{U})$ with projective $\bar{U}$.
(iii) For any functor in (2.2.1) its source is a base for the h-topology of the target, and the induced topology on the source is the h-topology.
(iv) The functors in (2.2.2) are continuous for the h-topologies.

[11] In fact, every V sits in a K-pair due to Nagata's theorem.

2.6. By §2.1 and §2.5, ϕ identifies h-sheaves on $\mathcal{V}ar_K$, resp. $\mathcal{V}ar_{\bar{K}}$, with h-sheaves on $\mathcal{V}ar_K^{\mathrm{c}}$, $\mathcal{V}ar_K^{\mathrm{nc}}$, $\mathcal{V}ar_K^{\mathrm{cc}}$, $\mathcal{V}ar_K^{\mathrm{ss}}$, resp. $\mathcal{V}ar_{\bar{K}}^{\mathrm{cc}}$, $\mathcal{V}ar_{\bar{K}}^{\mathrm{ss}}$. Thus we have the *h-localization* functors

$$\mathcal{PS}h(\mathcal{V}ar_K^{?}) \to \mathcal{V}ar_{K\mathrm{h}}^{\widetilde{\ \ }}, \quad \mathcal{PS}h(\mathcal{V}ar_{\bar{K}}^{?}) \to \mathcal{V}ar_{\bar{K}\mathrm{h}}^{\widetilde{\ \ }} \tag{2.6.1}$$

which assign to any presheaf $\mathcal{F}$ on pairs the corresponding h-sheaf $\mathcal{F}^{\sim}$ viewed as an h-sheaf on varieties.

Remark. For any presheaf on $\mathcal{V}ar_K^{\mathrm{c}}$, $\mathcal{V}ar_K^{\mathrm{cc}}$ or $\mathcal{V}ar_{\bar{K}}^{\mathrm{cc}}$, its h-sheafification coincides with h-sheafification of its restriction to resp. $\mathcal{V}ar_K^{\mathrm{nc}}$, $\mathcal{V}ar_K^{\mathrm{ss}}$ or $\mathcal{V}ar_{\bar{K}}^{\mathrm{ss}}$. For a presheaf on $\mathcal{V}ar_{\bar{K}/K}^{\mathrm{ss}}$, its h-sheafification is the same as h-sheafification of its restriction to $\mathcal{V}ar_{\bar{K}/K'}^{\mathrm{ss}}$, where $K' \subset \bar{K}$ is any finite extension of K (see Remark in §2.2).

3. The p-adic period map

3.1. *The derived de Rham algebra in logarithmic setting.* We refer to [K1] for log scheme basics. There are two (in general, nonequivalent) ways to define the cotangent complex for log schemes due, respectively, to Gabber and Olsson; see [Ol].[12] Gabber's approach ([Ol, §8]) is more direct and precise;[13] we recall it briefly.

For a commutative ring A, a prelog structure on A is a homomorphism of monoids $\alpha : L \to A$, where L is a commutative integral monoid (written multiplicatively) and A is viewed as a monoid with respect to the product. Rings equipped with prelog structures form a category in an evident way; denote its objects simply by (A, L). For a fixed (A, L), let $\mathcal{C}_{(A,L)}$ be the category of morphisms $(A, L) \to (B, M)$; we denote such an object by $(B, M)/(A, L)$. Let $\Omega_{(B,M)/(A,L)}$ be the B-module of relative Kähler log differentials: it is generated by $\Omega_{B/A}$ and elements $d\log m$, $m \in M$, subject to relations $d\log(m_1 m_2) = d\log m_1 + d\log m_2$, $\alpha(m) d\log m = d\alpha(m)$, and $d\log m = 0$ if m is in the image of L. The de Rham dg algebra of relative log forms $\Omega^{\cdot}_{(B,M)/(A,L)}$ has components $\Omega^i_{(B,M)/(A,L)} := \Lambda^i_B \Omega_{(B,M)/(A,L)}$; elements $d\log m$ are degree 1 cycles. It carries the Hodge filtration $F^n = \Omega^{\geq n}_{(B,M)/(A,L)}$.

A pair of sets I, J yields a free object $P_{(A,L)}[I, J]$ in $\mathcal{C}_{(A,L)}$: the corresponding ring is a polynomial algebra $A[t_i, t_j]_{i \in I, j \in J}$, the monoid is $L \oplus \mathbb{N}[I]$, where $\mathbb{N}[I]$ is the free monoid generated by I, and the structure map sends the generator m_i of $\mathbb{N}[I]$, $i \in I$, to t_i. The de Rham algebra $\Omega^{\cdot}_{P_{(A,L)}[I,J]/(A,L)}$ is freely generated, as a graded commutative A-algebra, by elements t_i, t_j of degree 0 and $d\log t_i := d\log m_i$, dt_j of degree 1, where $i \in I$, $j \in J$.

Every $(B, M)/(A, L) \in \mathcal{C}_{(A,L)}$ admits a canonical simplicial resolution $P_{\cdot} = P_{(A,L)}(B, M)_{\cdot}$. This is a simplicial object of $\mathcal{C}_{(A,L)}$ augmented over the object $(B, M)/(A, L)$ and such that every P_i is a free object as above. Thus we have the simplicial dg algebra $\Omega^{\cdot}_{P_{\cdot}/(A,L)}$ filtered by the Hodge filtration F. Denote by $L\Omega^{\cdot}_{(B,M)/(A,L)}$ the corresponding total complex, $L\Omega^a_{(B,M)/(A,L)} = \bigoplus_{j-i=a} \Omega^j_{P_i/(A,L)}$; this is a filtered commutative dg algebra. Let $L\Omega^{\cdot}_{(\hat{B},M)/(A,L)}$ be its F-completion; as in §1.2, *we understand it as a mere projective system of quotients* $L\Omega^{\cdot}_{(\hat{B},M)/(A,L)}/F^n$. One has a natural quasi-isomorphism of graded dg algebras $\mathrm{gr}^*_F L\Omega^{\cdot}_{(\hat{B},M)/(A,L)} \xrightarrow{\sim} (L\Lambda^*_B(\mathrm{L}_{(B,M)/(A,L)}))[-*]$. Here $\mathrm{L}_{(B,M)/(A,L)} := \Omega_{P_{\cdot}/(A,L)} \otimes_{P_{\cdot}} B$ is the relative

[12] In all situations that we will consider, the two versions coincide by [Ol, 8.34].

[13] It produces a true complex, while Olsson's construction yields a mere compatible datum of the canonical filtration truncations.

log cotangent complex; it is acyclic in positive degrees, and $H^0\mathrm{L}_{(B,M)/(A,L)} = \Omega_{(B,M)/(A,L)}$. The constructions are compatible with direct limits. If in the above definition we replace $P.$ by any free simplicial resolution of $(B,M)/(A,L)$, then the output is naturally quasi-isomorphic to $L\Omega^{\cdot}_{(\hat{B},M)/(A,L)}$. The plain cotangent complex and derived de Rham algebra for B/A map naturally to logarithmic ones.

For any map $(X,\mathcal{M}) \to (S,\mathcal{L})$ of integral log schemes, the above construction, being étale sheafified, yields the log cotangent complex $\mathrm{L}_{(X,\mathcal{M})/(S,\mathcal{L})}$, the derived log de Rham algebra $L\Omega^{\cdot}_{(X,\mathcal{M})/(S,\mathcal{L})}$, and its F-completion $L\Omega^{\cdot}_{(\hat{X},\mathcal{M})/(S,\mathcal{L})}$, which are complexes of sheaves on $X_{\text{ét}}$. We use only the completed complex $L\Omega^{\cdot}\hat{}$.

3.2. Let $(U,\bar{U})$ be a pair as in §2.2. We view $\bar{U}$ as a log scheme with the usual integral log structure $\mathcal{O}_{\bar{U}} \cap j_*\mathcal{O}_U^\times \to \mathcal{O}_{\bar{U}}$; by abuse of notation, let us denote this log scheme again by $(U,\bar{U})$. Any morphism of pairs $(U,\bar{U}) \to (V,\bar{V})$ is a morphism of log schemes, so we have the relative log cotangent complex $\mathrm{L}_{(U,\bar{U})/(V,\bar{V})}$, the derived log de Rham algebra $L\Omega^{\cdot}_{(\hat{U},\bar{U})/(V,\bar{V})}$, etc., as above. There is a canonical morphism $L\Omega^{\cdot}\hat{}_{\bar{U}/\bar{V}} \to L\Omega^{\cdot}_{(\hat{U},\bar{U})/(V,\bar{V})}$. We also have "absolute" complexes: in the arithmetic K- or $\bar{K}$-setting, these are $\mathrm{L}_{(U,\bar{U})} := \mathrm{L}_{(U,\bar{U})/O_K}$, $L\Omega^{\cdot}_{(\hat{U},\bar{U})} := L\Omega^{\cdot}_{(\hat{U},\bar{U})/O_K}$, where O_K is considered with the trivial log structure $O_K^\times$; for the geometric K- or $\bar{K}$-setting, replace O_K by K, resp. $\bar{K}$.

Remark. For $(V,\bar{V}) \in \mathcal{V}ar^{\mathrm{nc}}_{\bar{K}}$ one has $L\Omega^{\cdot}_{(\hat{V},\bar{V})} \xrightarrow{\sim} \Omega^{\cdot}_{(V,\bar{V})}$. Hence for $(U,\bar{U}) \in \mathcal{V}ar^{\mathrm{ss}}_{\bar{K}}$ one has $R\Gamma(\bar{U}, L\Omega^{\cdot}_{(\hat{U},\bar{U})}) \otimes \mathbb{Q} \xrightarrow{\sim} R\Gamma(\bar{U}_{\bar{K}}, \Omega^{\cdot}_{(U,\bar{U}_{\bar{K}})})$. Ditto for pairs over K.

Consider now the arithmetic $\bar{K}$-pair $\mathrm{Spec}(\bar{K}, O_{\bar{K}}) := (\mathrm{Spec}\,\bar{K}, \mathrm{Spec}\,O_{\bar{K}})$:

Lemma. *The cotangent complex $\mathrm{L}_{(\bar{K},O_{\bar{K}})}$ is acyclic in nonzero degrees, and the canonical map $\Omega_{O_{\bar{K}}} \to \Omega_{(\bar{K},O_{\bar{K}})} := H^0L_{(\bar{K},O_{\bar{K}})}$ is an isomorphism. Therefore the canonical map $A_{dR} := L\Omega^{\cdot}\hat{}_{O_{\bar{K}}} \to L\Omega^{\cdot}\hat{}_{(\bar{K},O_{\bar{K}})}$ is a filtered quasi-isomorphism.*

Proof. For a finite extension K' of K consider the log scheme $\mathrm{Spec}(K', O_{K'}) := (\mathrm{Spec}\,K', \mathrm{Spec}\,O_{K'})$. It is a log complete intersection over O_K (see [Ol], 6.8). If π is a generator of $O_{K'}/O_K$, $f(t)$ its minimal polynomial, then, by [Ol, 6.9], $\mathrm{L}_{(K',O_{K'})}$ is quasi-isomorphic to the cone of the multiplication by $f'(\pi)$ map $O_{K'} \to O_{K'} \subset \mathfrak{m}_{K'}^{-1}$. Thus $\mathrm{L}_{(K',O_{K'})}$ is acyclic in nonzero degrees, $\Omega_{(K',O_{K'})} := H^0\mathrm{L}_{(K',O_{K'})}$ is a cyclic $O_{K'}$-module, and the canonical map $\Omega_{O_{K'}} \to \Omega_{(K',O_{K'})}$ is an embedding with cokernel isomorphic to the residue field $O_{K'}/\mathfrak{m}_{K'}$. Now pass to the inductive limit, and use the fact that $\Omega_{O_{\bar{K}}}$ is p-divisible (see §1.3). □

3.3. Consider the presheaf $(U,\bar{U}) \mapsto R\Gamma^{\natural}_{\mathrm{dR}}(U,\bar{U}) := R\Gamma(\bar{U}, L\Omega^{\cdot}_{(\hat{U},\bar{U})})$ of filtered E_∞ dg O_K-algebras on $\mathcal{V}ar^{\mathrm{ss}}_{\bar{K}}$. Denote by $\mathcal{A}^{\natural}_{\mathrm{dR}}$ its h-sheafification (2.6.1); this is an h-sheaf of filtered E_∞ O_K-algebras on $\mathcal{V}ar_{\bar{K}}$ (as above, we see it as the projective system of quotients modulo F^i). Since $\mathrm{A}_{\mathrm{dR}} = \mathcal{A}^{\natural}_{\mathrm{dR}}(\mathrm{Spec}\,\bar{K})$ by Lemma in §3.2, A_{dR}, viewed as a constant filtered h-sheaf, maps into $\mathcal{A}^{\natural}_{\mathrm{dR}}$.

Theorem (p-adic Poincaré lemma)**.** *The maps $A_{dR} \otimes^L \mathbb{Z}/p^n \to \mathcal{A}^{\natural}_{dR} \otimes^L \mathbb{Z}/p^n$ are filtered quasi-isomorphisms of h-sheaves on $\mathcal{V}ar_{\bar{K}}$.*

For a proof, see §4. Assuming it, let us define the p-adic period map ρ.

3.4. *The Hodge-Deligne filtration.* For this subsection, K is any field of characteristic 0. Consider the presheaf $(V,\bar V)\mapsto R\Gamma_{\mathrm{dR}}(V,\bar V):=R\Gamma(\bar V,\Omega^{\cdot}_{(V,\bar V)})$ of filtered E_∞ dg K-algebras on $\mathcal{V}ar_K^{\mathrm{nc}}$. Let $\mathcal{A}_{\mathrm{dR}}$ be its h-sheafification (2.6.1), which is an h-sheaf of filtered E_∞ K-algebras on $\mathcal{V}ar_K$ (viewed as the projective system of quotients modulo F^i). For any $X\in\mathcal{V}ar_K$ set

$$R\Gamma_{\mathrm{dR}}(X):=R\Gamma(X_{\mathrm{h}},\mathcal{A}_{\mathrm{dR}}). \tag{3.4.1}$$

This is Deligne's de Rham complex of X equipped with Deligne's Hodge filtration.

Proposition. *(i) For $(V,\bar V)\in\mathcal{V}ar_K^{nc}$ the canonical map $R\Gamma_{dR}(V,\bar V)\to R\Gamma_{dR}(V)$ is a filtered quasi-isomorphism.*
(ii) The differential of $R\Gamma_{dR}(X)$ is strictly compatible with the filtration. $H^i_{dR}(X):=H^iR\Gamma_{dR}(X)$ are K-vector spaces of dimension equal to $\dim H^i_{\acute{e}t}(X_{\bar K},\mathbb{Q}_p)$.
(iii) For any smooth variety X there is a canonical (nonfiltered) quasi-isomorphism $R\Gamma(X,\Omega^{\cdot}_X)\xrightarrow{\sim}R\Gamma_{dR}(X)$.

Proof. By Lefschetz's principle, we can assume that $K=\mathbb{C}$. For $(V,\bar V)\in\mathcal{V}ar_{\mathbb{C}}^{\mathrm{nc}}$ the maps $R\Gamma_{\mathrm{dR}}(V,\bar V)\to R\Gamma_{\mathrm{dR}}(V,\Omega^{\cdot}_V)\to R\Gamma(V_{\mathrm{cl}},\mathbb{C})$ are quasi-isomorphisms by [Gr]. Thus for any h-hypercovering[14] $(Y_\cdot,\bar Y_\cdot)/X$ of X in $\mathcal{V}ar_K^{\mathrm{nc}}$ the cohomological descent (see Remark in §2.4) yields a canonical quasi-isomorphism $R\Gamma(\bar Y_\cdot,\Omega^{\cdot}_{(Y_\cdot,\bar Y_\cdot)})\xrightarrow{\sim}R\Gamma(X_{\mathrm{cl}},\mathbb{C})$. If we equip $R\Gamma(X_{\mathrm{cl}},\mathbb{C})$ with the Hodge-Deligne filtration of mixed Hodge theory [D], then this is a filtered quasi-isomorphism. Therefore we have a canonical filtered quasi-isomorphism $R\Gamma_{\mathrm{dR}}(X)\xrightarrow{\sim}R\Gamma(X_{\mathrm{cl}},\mathbb{C})$. Now (i) and the second assertion of (ii) are clear; the first assertion of (ii) follows from mixed Hodge theory. The quasi-isomorphism in (iii) is $R\Gamma(X,\Omega^{\cdot}_X)\xrightarrow{\sim}R\Gamma(Y_\cdot,\Omega^{\cdot}_{Y_\cdot})\xleftarrow{\sim}R\Gamma(\bar Y_\cdot,\Omega^{\cdot}_{(Y_\cdot,\bar Y_\cdot)})$, where the arrows are quasi-isomorphisms by the cohomological descent (since $R\Gamma(X,\Omega^{\cdot}_X)\xrightarrow{\sim}R\Gamma(X_{\mathrm{cl}},\mathbb{C})$). □

3.5. We return to the setting of §3.3, so K is our p-adic field. Let X be any variety over $\bar K$. It yields a filtered E_∞ O_K-algebra

$$R\Gamma^{\natural}_{\mathrm{dR}}(X):=R\Gamma(X_{\mathrm{h}},\mathcal{A}^{\natural}_{\mathrm{dR}}). \tag{3.5.1}$$

Since $\mathrm{A}_{\mathrm{dR}}\otimes\mathbb{Q}=\bar K$ (see Remark (i) in §1.5), $R\Gamma^{\natural}_{\mathrm{dR}}(X)\otimes\mathbb{Q}$ is a $\bar K$-algebra. By Remark in §3.2, we have a filtered quasi-isomorphism of E_∞ $\bar K$-algebras

$$R\Gamma^{\natural}_{\mathrm{dR}}(X)\otimes\mathbb{Q}\xrightarrow{\sim}R\Gamma_{\mathrm{dR}}(X). \tag{3.5.2}$$

Let us compute $R\Gamma^{\natural}_{\mathrm{dR}}(X)\widehat{\otimes}\mathbb{Z}_p$. Consider the morphisms of filtered complexes $R\Gamma(X_{\text{ét}},\mathbb{Z})\otimes^L\mathrm{A}_{\mathrm{dR}}\xrightarrow{\sim}R\Gamma(X_{\text{ét}},\mathrm{A}_{\mathrm{dR}})\to R\Gamma(X_{\mathrm{h}},\mathrm{A}_{\mathrm{dR}})\to R\Gamma(X_{\mathrm{h}},\mathcal{A}^{\natural}_{\mathrm{dR}})=R\Gamma^{\natural}_{\mathrm{dR}}(X)$. After applying $\cdot\otimes^L\mathbb{Z}/p^n$, the arrows become filtered quasi-isomorphisms (the first one by Remark in §2.4, the second one by the Poincaré lemma in §3.3), so we get a filtered quasi-isomorphism $R\Gamma(X_{\text{ét}},\mathbb{Z}/p^n)\otimes^L\mathrm{A}_{\mathrm{dR}}\xrightarrow{\sim}R\Gamma^{\natural}_{\mathrm{dR}}(X)\otimes^L\mathbb{Z}/p^n$. Since $R\Gamma_{\text{ét}}(X,\mathbb{Z}_p)=\mathrm{holim}R\Gamma(X_{\text{ét}},\mathbb{Z}/p^n)=R\Gamma(X_{\text{ét}},\mathbb{Z})\widehat{\otimes}\mathbb{Z}_p$ (see §1.1) is a perfect $\mathbb{Z}_p$-complex and $R\Gamma(X_{\text{ét}},\mathbb{Z}/p^n)=R\Gamma_{\text{ét}}(X,\mathbb{Z}_p)\otimes^L_{\mathbb{Z}_p}\mathbb{Z}/p^n$, one has, passing to the homotopy limit as in §1.1, $R\Gamma_{\text{ét}}(X,\mathbb{Z}_p)\otimes^L_{\mathbb{Z}_p}(\mathrm{A}_{\mathrm{dR}}\widehat{\otimes}\mathbb{Z}_p)\xrightarrow{\sim}R\Gamma^{\natural}_{\mathrm{dR}}(X)\widehat{\otimes}\mathbb{Z}_p$. Tensoring by $\mathbb{Q}$, we get a filtered quasi-isomorphism of filtered E_∞ $\mathrm{B}^+_{\mathrm{dR}}$-algebras (see (1.5.1))

$$\beta:R\Gamma_{\text{ét}}(X,\mathbb{Q}_p)\otimes\mathrm{B}^+_{\mathrm{dR}}\xrightarrow{\sim}R\Gamma^{\natural}_{\mathrm{dR}}(X)\widehat{\otimes}\mathbb{Q}_p. \tag{3.5.3}$$

[14] Here we view X as an h-sheaf on $\mathcal{V}ar_K^{\mathrm{nc}}$, so $(Y_\cdot,\bar Y_\cdot)$ is a simplicial object of $\mathcal{V}ar_K^{\mathrm{nc}}$ equipped with an augmentation map $Y_\cdot\to X$ that makes $Y_\cdot$ an h-hypercovering of X.

Let $\alpha : R\Gamma_{\mathrm{dR}}(X) \otimes_{\bar{K}} \mathrm{B}^+_{\mathrm{dR}} \to R\Gamma^\natural_{\mathrm{dR}}(X)\widehat{\otimes}\mathbb{Q}_p$ be the $\mathrm{B}^+_{\mathrm{dR}}$-linear extension of the composition $R\Gamma_{\mathrm{dR}}(X) \xrightarrow{\sim} R\Gamma^\natural_{\mathrm{dR}}(X) \otimes \mathbb{Q} \to R\Gamma^\natural_{\mathrm{dR}}(X)\widehat{\otimes}\mathbb{Q}_p$, where the first arrow is inverse to (3.5.2) and the second one comes from the canonical map $? \to ?\widehat{\otimes}\mathbb{Z}_p$. We get a morphism of filtered E_∞ $\mathrm{B}^+_{\mathrm{dR}}$-algebras

$$\rho = \rho_{\mathrm{dR}} := \beta^{-1}\alpha : R\Gamma_{\mathrm{dR}}(X) \otimes_{\bar{K}} \mathrm{B}^+_{\mathrm{dR}} \to R\Gamma_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}^+_{\mathrm{dR}}. \tag{3.5.4}$$

Remarks. (i) The Galois group $\mathrm{Gal}(\bar{K}/K)$ acts on $\mathcal{V}ar_{\bar{K}\mathrm{h}}$ and on both sides of (3.5.4) by transport of structure, and ρ^+ is evidently compatible with this action. In particular, if X is defined over K, i.e., $X = X_K \otimes_K \bar{K}$, then $R\Gamma_{\mathrm{dR}}(X) = R\Gamma_{\mathrm{dR}}(X_K) \otimes_K \bar{K}$, and we can rewrite (3.5.4) as a $\mathrm{Gal}(\bar{K}/K)$-equivariant morphism

$$\rho : R\Gamma_{\mathrm{dR}}(X_K) \otimes_K \mathrm{B}^+_{\mathrm{dR}} \to R\Gamma_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}^+_{\mathrm{dR}}. \tag{3.5.5}$$

(ii) The map ρ does not change if we replace K by any of its finite extensions that are contained in $\bar{K}$ (see Remark in §2.6).

3.6. **Theorem.** *The B_{dR}-linear extension of ρ is a filtered quasi-isomorphism: for any $X \in \mathcal{V}ar_{\bar{K}}$ one has*

$$\rho : R\Gamma_{dR}(X) \otimes_{\bar{K}} B_{dR} \xrightarrow{\sim} R\Gamma_{\acute{e}t}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR}. \tag{3.6.1}$$

Proof. (a) *The case of $X = \mathbb{G}_m = \mathbb{G}_{m\bar{K}}$*: The $\bar{K}$-line $H^1_{\mathrm{dR}}(\mathbb{G}_m) = \mathrm{gr}^1_F H^1_{\mathrm{dR}}(\mathbb{G}_m)$ is generated by $d\log t$. The $\mathbb{Z}_p$-line $H^1_{\text{ét}}(\mathbb{G}_m, \mathbb{Z}_p)(1) = H^1_{\text{ét}}(\mathbb{G}_m, \mathbb{Z}_p(1))$ is generated by the class $cl(\mathfrak{k})$ of the Kummer $\mathbb{Z}_p(1)$-torsor $\mathfrak{k} = \varprojlim \mathfrak{k}_n$, $\mathfrak{k}_n := (t^{1/p^n})$. Due to the canonical identification $\mathbb{C}_p(1) \xrightarrow{\sim} \mathfrak{m}_{\mathrm{dR}}/\mathfrak{m}^2_{\mathrm{dR}} = \mathrm{gr}^1_F \mathrm{B}_{\mathrm{dR}}$, see §1.4, §1.5, we can view $cl(\mathfrak{k})$ as a generator of the $\mathbb{C}_p$-line $H^1_{\text{ét}}(\mathbb{G}_m, \mathbb{Q}_p) \otimes \mathrm{gr}^1_F \mathrm{B}_{\mathrm{dR}}$.

Lemma. *One has $gr^1_F(\rho)(d\log t) = cl(\mathfrak{k}) \in H^1_{\acute{e}t}(\mathbb{G}_m, \mathbb{Q}_p) \otimes gr^1_F B_{dR}$.*

Proof of Lemma. We make a mod p^n computation. Consider the ss-pair $(\mathbb{G}_m, \bar{\mathbb{G}}_m)$, $\bar{\mathbb{G}}_m := \mathbb{P}^1_{O_{\bar{K}}}$. One has $\mathrm{gr}^1_F L\Omega^\cdot_{(\hat{\mathbb{G}}_m, \bar{\mathbb{G}}_m)} = \Omega^1_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)}[-1]$, so $d\log t \in \Gamma(\bar{\mathbb{G}}_m, \Omega^1_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)})$ is a 1-cocycle in $\mathrm{gr}^1_F R\Gamma^\natural_{\mathrm{dR}}(\mathbb{G}_m)$. As in §1.1, set $C_n := \mathcal{C}one\,(p^n : \mathbb{Z} \to \mathbb{Z})$. Let $d\log cl(\mathfrak{k}_n)$ be the image of the class $cl(\mathfrak{k}_n)$ of $\mathfrak{k}_n$ by the composition $H^1(\mathbb{G}_{m\,\text{ét}}, \mu_{p^n}) \to H^1(\mathbb{G}_{m\,\text{ét}}, \mathrm{gr}^1_F \mathrm{A}_{\mathrm{dR}} \otimes C_n) \to \mathrm{gr}^1_F R\Gamma^\natural_{\mathrm{dR}}(\mathbb{G}_m) \otimes C_n$, where the first arrow comes from the coefficient maps $\mu_{p^n} \xrightarrow{d\log} \Omega_{O_{\bar{K}}\, p^n} \hookrightarrow \Omega_{O_{\bar{K}}}[-1] \otimes C_n = \mathrm{gr}^1_F \mathrm{A}_{\mathrm{dR}} \otimes C_n$. To prove the lemma, we will show that the image of $d\log t$ by the embedding $\mathrm{gr}^1_F R\Gamma^\natural_{\mathrm{dR}}(\mathbb{G}_m) \hookrightarrow \mathrm{gr}^1_F R\Gamma^\natural_{\mathrm{dR}}(\mathbb{G}_m) \otimes C_n$ is homologous to $d\log cl(\mathfrak{k}_n)$.

Let $\mathbb{G}_m^{\tilde{}}$ be a copy of $\mathbb{G}_m$ with parameter $\tilde{t}$, and $\pi : \mathbb{G}_m^{\tilde{}} \to \mathbb{G}_m$ be the projection $t = \tilde{t}^{p^n}$. Thus $\mathbb{G}_m^{\tilde{}}/\mathbb{G}_m$ is our μ_{p^n}-torsor $\mathfrak{k}_n$, so $cl(\mathfrak{k}_n)$ is represented by a Čech μ_{p^n}-cocycle $c(\mathfrak{k}_n)$ for the étale covering $\mathbb{G}_m^{\tilde{}}/\mathbb{G}_m$. The corresponding Čech hypercovering is the twist of $\mathbb{G}_m^{\tilde{}}$ by the universal μ_{p^n}-torsor $\mathfrak{t}_n$ over the classifying simplicial space $B_{\mu_{p^n}\cdot}$, so for any sheaf $\mathcal{F}$ the Čech complex of $\mathbb{G}_m^{\tilde{}}/\mathbb{G}_m$ with coefficients in $\mathcal{F}$ is the cochain complex $C^\cdot(\mu_{p^n}, \Gamma(\mathbb{G}_m^{\tilde{}}, \mathcal{F}))$ for μ_{p^n} acting on sections by the translations. The 1-cocycle $c(\mathfrak{k}_n)$ is the identity map $\mu_{p^n} \to \mu_{p^n} = \Gamma(\mathbb{G}_m^{\tilde{}}, \mu_{p^n})$.

Our π extends to the h-covering of semi-stable pairs $(\mathbb{G}_m^{\tilde{}}, \bar{\mathbb{G}}_m^{\tilde{}}) \to (\mathbb{G}_m, \bar{\mathbb{G}}_m)$, and the Čech hypercovering extends to a hypercovering in $\mathcal{V}ar^{\mathrm{ss}}_{\bar{K}}$ which is the $\mathfrak{t}_n$-twist of $(\mathbb{G}_m^{\tilde{}}, \bar{\mathbb{G}}_m^{\tilde{}})$. So one has a canonical map $C^\cdot(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m^{\tilde{}}, \Omega^1_{(\mathbb{G}_m^{\tilde{}}, \bar{\mathbb{G}}_m^{\tilde{}})}))[-1] \to \mathrm{gr}^1_F R\Gamma^\natural_{\mathrm{dR}}(\mathbb{G}_m)$; hence $C^\cdot(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m^{\tilde{}}, \Omega^1_{(\mathbb{G}_m^{\tilde{}}, \bar{\mathbb{G}}_m^{\tilde{}})}))[-1] \otimes C_n \to \mathrm{gr}^1_F R\Gamma^\natural_{\mathrm{dR}}(\mathbb{G}_m) \otimes C_n$. Both $d\log t$ and $d\log c(\mathfrak{k}_n)$ are 1-cocycles in $C^\cdot(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m^{\tilde{}}, \Omega^1_{(\mathbb{G}_m^{\tilde{}}, \bar{\mathbb{G}}_m^{\tilde{}})}))[-1] \otimes C_n$: namely, $d\log t \in C^0(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m^{\tilde{}}, \Omega^1_{(\mathbb{G}_m^{\tilde{}}, \bar{\mathbb{G}}_m^{\tilde{}})})[-1])$ and $d\log c(\mathfrak{k}_n) \in C^1(\mu_{p^n}, \Omega_{O_{\bar{K}}\, p^n})$

$\subset C^1(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m^{\sim}, \Omega^1_{(\mathbb{G}_m^{\sim}, \bar{\mathbb{G}}_m^{\sim})})[-1] \otimes C_n)$. Their difference is the differential of the 0-cochain $d\log\tilde{t} \in C^0(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m^{\sim}, \Omega^1_{(\mathbb{G}_m^{\sim}, \bar{\mathbb{G}}_m^{\sim})})) \subset C^0(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m^{\sim}, \Omega^1_{(\mathbb{G}_m^{\sim}, \bar{\mathbb{G}}_m^{\sim})})[-1] \otimes C_n)$, q.e.d. □

We see that for $X = \mathbb{G}_m$ the map ρ of (3.6.1) is a filtered quasi-isomorphism. It provides a canonical generator $\rho(d\log t)/cl(\mathfrak{k})$ of $\mathfrak{m}_{\mathrm{dR}}(-1)$. Thus we have a canonical identification between $H^{\cdot}_{\text{ét}}(X, \mathbb{Q}_p(n)) \otimes \mathrm{B}_{\mathrm{dR}}$ and $H^{\cdot}_{\text{ét}}(X, \mathbb{Q}_p) \otimes \mathrm{B}_{\mathrm{dR}}$ with the filtration shifted by n, etc.

Remark. The above generator is equal to Fontaine's generator from [F3, 1.5.4]. Indeed, they coincide modulo $\mathfrak{m}^2_{\mathrm{dR}}(-1)$ by the lemma, and both are $\mathrm{Gal}(\bar{K}/K)$-invariant (see Remark in §3.5). Since $H^0(\mathrm{Gal}(\bar{K}/K), \mathfrak{m}^2_{\mathrm{dR}}(-1)) = 0$, we are done.

(b) *Compatibility with the Gysin maps*: Let $i : Y \hookrightarrow X$ be a closed codimension 1 embedding of smooth varieties. It yields the Gysin isomorphisms $i_{*\mathrm{dR}} : R\Gamma_{\mathrm{dR}}(Y) \overset{\sim}{\to} R\Gamma_{\mathrm{dR}\,Y}(X)(1)[2] := \mathcal{C}one(R\Gamma_{\mathrm{dR}}(X) \to \Gamma_{\mathrm{dR}}(X \smallsetminus Y))(1)[1]$, $i_{*\mathbb{Q}_p} : R\Gamma_{\text{ét}}(Y, \mathbb{Q}_p) \overset{\sim}{\to} R\Gamma_{\text{ét}\,Y}(X, \mathbb{Q}_p)(1)[2]$. Let us show that ρ commutes with the Gysin maps.

Consider the deformation to the normal cone diagram

$$\begin{array}{ccccc} \mathcal{L} & \hookrightarrow & X_{\mathbb{A}^1}^{\sim} & \hookleftarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ Y & \hookrightarrow & Y_{\mathbb{A}^1} & \hookleftarrow & Y. \end{array} \tag{3.6.2}$$

Here $Y_{\mathbb{A}^1} = Y \times \mathbb{A}^1$, $X_{\mathbb{A}^1}^{\sim}$ is $X \times \mathbb{A}^1$ with $Y \times \{0\}$ blown up, the left arrow is the zero section of the normal bundle $\mathcal{L}$ to Y in X, and the bottom embeddings are $y \mapsto (y, 0), (y, 1)$. It yields a commutative diagram of the de Rham cohomology

$$\begin{array}{ccccc} R\Gamma_{\mathrm{dR}\,Y}(\mathcal{L})(1)[2] & \leftarrow & R\Gamma_{\mathrm{dR}\,Y_{\mathbb{A}^1}}(X_{\mathbb{A}^1}^{\sim})(1)[2] & \to & R\Gamma_{\mathrm{dR}\,Y}(X)(1)[2] \\ \uparrow & & \uparrow & & \uparrow \\ R\Gamma_{\mathrm{dR}}(Y) & \leftarrow & R\Gamma_{\mathrm{dR}}(Y_{\mathbb{A}^1}) & \to & R\Gamma_{\mathrm{dR}}(Y), \end{array} \tag{3.6.3}$$

where the vertical arrows are the Gysin isomorphisms and the horizontal ones are pullbacks. There is a similar diagram for the $\mathbb{Q}_p$-cohomology. The horizontal maps are filtered quasi-isomorphisms, so, since ρ is compatible with pullbacks, we see that the Gysin compatibility for $Y \hookrightarrow X$ amounts to one for $Y \hookrightarrow \mathcal{L}$.

So we can assume that X is a line bundle $\mathcal{L}$ over Y and i its zero section. Now the source of both i_*'s are dg algebras, the targets are modules over them (due to the projection $\mathcal{L} \to Y$), and i_*'s are morphisms of modules. Thus it suffices to check that ρ identifies the images of 1. The assertion is local with respect to Y; hence we can assume that $\mathcal{L}$ is trivial. By base change, we reduced to the case when Y is a point, where we are done by (a).

(c) *The case of a smooth projective* X: Let us check that the morphism of bigraded rings $\mathrm{gr}^{\cdot}_F \rho^* : \mathrm{gr}^{\cdot}_F H^*_{\mathrm{dR}}(X) \otimes_{\bar{K}} \mathrm{gr}^{\cdot}_F \mathrm{B}_{\mathrm{dR}} \to H^*_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{gr}^{\cdot}_F \mathrm{B}_{\mathrm{dR}}$ is an isomorphism. It is an isomorphism for $* = 0$. By (b), $\mathrm{gr}^1_F \rho^2$ identifies the classes c of a hyperplane section. Since the product with $c^{\dim X}$ identifies H^0 and $H^{2\dim X}$, $\mathrm{gr}^{\cdot}_F \rho^{2\dim X}$ is an isomorphism. Therefore, since $\mathrm{gr}^{\cdot}_F \rho^*$ is compatible with the Poincaré pairing for classes of opposite degrees and the latter is nondegenerate, $\mathrm{gr}^{\cdot}_F \rho^*$ is injective. Since $\dim_{\bar{K}} H^*_{\mathrm{dR}}(X) = \dim_{\mathbb{Q}_p} H^*_{\text{ét}}(X, \mathbb{Q}_p)$, we are done.

(d) *The case of $X = \bar{X} \smallsetminus D$, where $\bar{X}$ is smooth projective, D is a strict normal crossings divisor*: Let Y be an irreducible component of D, D' be the union of the other components; set $X' := \bar{X} \smallsetminus D'$, $Y' := Y \smallsetminus D'$. By induction by the number of components D (starting with (c)), we can assume that the theorem holds for X' and Y'. By (b), ρ provides a morphism between the exact Gysin triangles for (Y', X'). It is a filtered quasi-isomorphism on the X' and Y' terms; hence it is a filtered quasi-isomorphism on the X term, q.e.d.

(e) *The case of arbitrary* X: If $Y_{\cdot}/X$ is any h-hypercovering of X, then the canonical map $R\Gamma_{\mathrm{dR}}(X) \to R\Gamma_{\mathrm{dR}}(Y_{\cdot})$ (which is the total complex of the cosimplicial system of filtered complexes $R\Gamma_{\mathrm{dR}}(Y_i)$) is a filtered quasi-isomorphism by the construction of $R\Gamma_{\mathrm{dR}}$, and $R\Gamma_{\text{ét}}(X, \mathbb{Q}_p) \xrightarrow{\sim} R\Gamma_{\text{ét}}(Y_{\cdot}, \mathbb{Q}_p)$ by cohomological descent (see Remark in 2.4). Thus if ρ is a filtered quasi-isomorphism for every Y_i, then it is a filtered quasi-isomorphism for X. We are done, since, by de Jong (or Hironaka), one can find $Y_{\cdot}/X$ with Y_i as in (d). $\square$

Remark. ρ is compatible with Chern classes of vector bundles: Indeed, $c_i(E)$ are determined in the usual way by $c_1(\mathcal{O}(1)_{\mathbb{P}(E)})$, so it suffices to show that ρ identifies c_1's of line bundles. Notice that the construction of ρ extends tautologically to simplicial schemes. By (a) above, ρ identifies the de Rham and étale Chern classes of the universal line bundle over the classifying simplicial scheme $B_{\mathbb{G}_m\cdot}$. For a line bundle $\mathcal{L}$ on X, choose a finite open covering $\{U_i\}$ of X such that $\mathcal{L}$ is trivial on U_i; let $\pi : X_{\cdot}^{\sim} \to X$ be the Čech hypercovering. Since π yields an isomorphism between the cohomology, it suffices to check that ρ identifies the Chern classes of $\pi^*\mathcal{L}$. This is true since $\pi^*\mathcal{L}$ is the pullback of the universal line bundle by a map $X_{\cdot}^{\sim} \to B_{\mathbb{G}_m\cdot}$.

4. Proof of the Poincaré lemma

4.1. Pick any $(V, \bar{V}) \in \mathcal{V}ar^{\mathrm{ss}}_{\bar{K}}$.

Proposition. *One has* $\mathrm{L}_{(V,\bar{V})} \xrightarrow{\sim} \Omega_{(V,\bar{V})}$, *the* $\mathcal{O}_{\bar{V}}$*-module* $\Omega_{\langle V,\bar{V}\rangle} := \Omega_{(V,\bar{V})/(\bar{K},O_{\bar{K}})}$ *is locally free of finite rank, and there is a canonical short exact sequence*

$$0 \to \mathcal{O}_{\bar{V}} \otimes_{O_{\bar{K}}} \Omega_{O_{\bar{K}}} \to \Omega_{(V,\bar{V})} \to \Omega_{\langle V,\bar{V}\rangle} \to 0. \tag{4.1.1}$$

Proof. We can assume that V is connected, so $(V, \bar{V})$ is the base change of a semistable K-pair $(U, \bar{U})$ as in 2.2(c), i.e., $(V, \bar{V}) = (U_{\bar{K}}, \bar{U}_{O_{\bar{K}}})$. For any finite extension K' of K_U, consider an arithmetic K-pair $(U_{K'}, \bar{U}_{O_{K'}}) := (U \otimes_{K_U} K', \bar{U} \otimes_{O_{K_U}} O_{K'})$. Set $\Omega_{\langle U,\bar{U}\rangle} := \Omega_{(U,\bar{U})/(K_U,O_{K_U})}$, $\Omega_{\langle U_{K'},\bar{U}_{O_{K'}}\rangle} := \Omega_{(U_{K'},\bar{U}_{O_{K'}})/(K',O_{K'})}$.

Lemma. *The log scheme* $(U_{K'}, \bar{U}_{O_{K'}})$ *coincides with the pullback of* $(U, \bar{U})$ *by the map* $\mathrm{Spec}\,(K', O_{K'}) \to \mathrm{Spec}\,(K_U, O_{K_U})$ *in the category of log schemes.*

Assume the lemma for a moment. The map $(U, \bar{U}) \to \mathrm{Spec}\,(O_{K_U}, K_U)$ is log smooth and integral; by the lemma, $(U_{K'}, \bar{U}_{O_{K'}}) \to \mathrm{Spec}\,(O_{K'}, K')$ enjoys the same properties. So, by [Ol, 8.34], $\mathrm{L}_{(U_{K'},\bar{U}_{O_{K'}})/(K',O_{K'})} \xrightarrow{\sim} \Omega_{\langle U_{K'},\bar{U}_{O_{K'}}\rangle} = O_{K'} \otimes_{O_{K_U}} \Omega_{\langle U,\bar{U}\rangle}$, which is a locally free $\mathcal{O}_{\bar{U}_{O_{K'}}}$-module of finite rank. Since $\mathrm{L}_{(K',O_{K'})} \xrightarrow{\sim} \Omega_{(K',O_{K'})}$ (see the proof of Lemma in §3.2), the canonical exact triangle ([Ol, 8.18]) $\mathcal{O}_{\bar{U}_{O_{K'}}} \otimes_{O_{K'}} \mathrm{L}_{(K',O_{K'})} \to \mathrm{L}_{(U_{K'},\bar{U}_{O_{K'}})} \to \mathrm{L}_{(U_{K'},\bar{U}_{O_{K'}})/(K',O_{K'})}$ reduces to the short exact sequence $0 \to \mathcal{O}_{\bar{U}_{O_{K'}}} \otimes_{O_{K'}} \Omega_{(K',O_{K'})} \to \Omega_{(U_{K'},\bar{U}_{O_{K'}})} \to \Omega_{\langle U_{K'},\bar{U}_{O_{K'}}\rangle} \to 0$. Pass to the limit by all $K' \subset \bar{K}$ and use the lemma in §3.2; we are done. $\square$

Proof of Lemma. The underlying scheme of the pullback log scheme is $\bar{U}_{O_{K'}}$. Let us show that its log structure map $\mathcal{M} \to \mathcal{O}_{\bar{U}_{O_{K'}}} \cap j_*\mathcal{O}^{\times}_{U_{K'}}$ is an isomorphism. The assertion is étale local, so we can assume that $\bar{U}$ is étale over $\mathrm{Spec}\, O_{K_U}[t_a, t_b, t_c]/$

$(\Pi t_a - \pi_{K_U})$, where a, b, c are in finite sets A, B, C, π_{K_U} is a uniformizing parameter in O_{K_U}, and U is the subscheme where all t_a, t_b are invertible. The log structure of $(U, \bar{U})$ is fine with a chart $\mathbb{N}[A \sqcup B] \to \mathcal{O}_{\bar{U}}$, which sends generators m_a, m_b of $\mathbb{N}[A \sqcup B]$ to t_a, t_b. Therefore[15] $\bar{U}_{O_{K'}}$ is étale over $\operatorname{Spec} O_{K'}[t_a, t_b, t_c]/(\Pi t_a - \pi^e_{K'})$, where e is the ramification index of K'/K_U, $\pi_{K'}$ is a uniformizing parameter in $O_{K'}$, and the log structure $\mathcal{M}$ has a chart $M_{A,B} \to \mathcal{O}_{\bar{U}_{O_{K'}}}$, where $M_{A,B}$ is the quotient of $\mathbb{N}[A \sqcup B] \oplus \mathbb{N}$ modulo the relation $\Pi m_a = m_\pi^e$ (m_π is the generator of the last summand $\mathbb{N}$), the chart is $m_a, m_b, m_\pi \mapsto t_a, t_b, \pi_{K'}$. Consider an embedding $M_{A,B} \hookrightarrow M^w_{A,B} := e^{-1}\mathbb{N}[A] \oplus \mathbb{N}[B]$, $m_a, m_b, m_\pi \mapsto m_a, m_b, \Pi m_a^{1/e}$. Its image is formed by those $\Pi m_a^{n_a/e} \Pi m_b^{n_b}$, $n_a, n_b \in \mathbb{N}$, such that $n_a - n_{a'} \in e\mathbb{Z}$ for any $a, a' \in A$; thus $M_{A,B}$ is saturated. Now the log scheme $(\bar{U}_{O_{K'}}, \mathcal{M})$ is evidently log regular in the sense of [K2, 2.1]; hence $\mathcal{M} \xrightarrow{\sim} \mathcal{O}_{\bar{U}_{O_{K'}}} \cap j_* \mathcal{O}^\times_{U_{K'}}$ by [K2, 11.6], q.e.d.[16]

The reference to [K2] can be replaced by the next explicit argument: It suffices to show that the map of sheaves $\mathcal{M}/\mathcal{O}^\times_{\bar{U}_{O_{K'}}} \to (\mathcal{O}_{\bar{U}_{O_{K'}}} \cap j_* \mathcal{O}^\times_{U_{K'}})/\mathcal{O}^\times_{\bar{U}_{O_{K'}}}$ is an isomorphism. The r.h.s. is the sheaf $\mathcal{D}$ of effective Cartier divisors supported on $\bar{U}_{O_{K'}} \smallsetminus U_{K'}$. Let $\mathcal{D}^w \supset \mathcal{D}$ be the sheaf of the corresponding effective Weil divisors. For $x \in \bar{U}_{O_{K'}}$, the fiber $(\mathcal{M}/\mathcal{O}^\times_{\bar{U}_{O_{K'}}})_x$ is the quotient M_{A_x, B_x} of $M_{A,B}$, where $A_x \subset A$, $B_x \subset B$ consist of those a, b such that t_a, t_b vanish at x. The map $M_{A_x, B_x} \to \mathcal{D}_x$ extends to an isomorphism $M^w_{A_x, B_x} \xrightarrow{\sim} \mathcal{D}^w_x$, which identifies a generator $m_a^{1/e}$ with the *reduced* divisor D_a of t_a, m_b with $D_b := \operatorname{div}(t_b)$. Thus $M_{A_x, B_x} \hookrightarrow \mathcal{D}_x$. To show that $\hookrightarrow$ is an isomorphism, we need to check that if $D = \Sigma n_a D_a + \Sigma n_b D_b$ is a Cartier divisor at x, then $n_a - n_{a'} \in e\mathbb{Z}$ for any $a, a' \in A_x$. We can assume that $A = \{a, a'\}$, $B = C = \emptyset$, so $\bar{U}_{O_{K'}}$ is a semi-stable curve over $O_{K'}$. The exceptional divisor of its minimal desingularization $\tilde{U}_{O_{K'}}$ is a chain of $e-1$ projective lines $P_1, \ldots, P_{e-1}$ with self-intersection indices $(P_i, P_i) = -2$. Let $\tilde{D} = n_a D_a^\flat + n_1 P_1 + \ldots + n_{e-1} P_{e-1} + n_{a'} D^\flat_{a'}$ be the pullback of D to $\tilde{U}_{O_{K'}}$; here $D_a^\flat$, $D^\flat_{a'}$ are strict transforms of D_a, $D_{a'}$. One has $(\tilde{D}, P_i) = 0$, i.e., $n_{i-1} - 2n_i + n_{i+1} = 0$ or $n_i - n_{i-1} = n_{i+1} - n_i$, where $n_0 := n_a$, $n_e := n_{a'}$. Thus $n_{a'} - n_a = e(n_1 - n_a) \in e\mathbb{Z}$, and we are done. $\square$

4.2. Set $\Omega^a_{\langle V, \bar{V} \rangle} := \Lambda^a_{\mathcal{O}_{\bar{V}}} \Omega_{\langle V, \bar{V} \rangle}$. Consider (4.1.1) as a 2-step filtration on $\Omega_{(V, \bar{V})}$; it splits locally since $\Omega_{\langle V, \bar{V} \rangle}$ is locally free. Passing to derived exterior powers, we get for any m a finite increasing filtration $I.$ on $\operatorname{gr}^m_F L\Omega^\cdot_{(\hat{V}, \bar{V})} = (L\Lambda^m_{\mathcal{O}_{\bar{V}}} \Omega_{(V, \bar{V})})[-m]$ with $\operatorname{gr}^I_a \operatorname{gr}^m_F L\Omega^\cdot_{(\hat{V}, \bar{V})} = \Omega^a_{\langle V, \bar{V} \rangle} \otimes_{O_{\bar{K}}} \operatorname{gr}^{m-a}_F \mathrm{A}_{\mathrm{dR}}[-a]$, hence on $\operatorname{gr}^m_F R\Gamma^\natural_{\mathrm{dR}}(V, \bar{V})$ with

$$\operatorname{gr}^I_a \operatorname{gr}^m_F R\Gamma^\natural_{\mathrm{dR}}(V, \bar{V}) = R\Gamma(\bar{V}, \Omega^a_{\langle V, \bar{V} \rangle}) \otimes^L_{O_{\bar{K}}} \operatorname{gr}^{m-a}_F \mathrm{A}_{\mathrm{dR}}[-a]. \tag{4.2.1}$$

Let $\mathcal{G}^a$ be the h-sheafification (see (2.6.1)) of the complex of presheaves $(V, \bar{V}) \mapsto R\Gamma(\bar{V}, \Omega^a_{\langle V, \bar{V} \rangle})$ on $\mathcal{V}ar^{\mathrm{ss}}_{\bar{K}}$. This is a complex of h-sheaves of $O_{\bar{K}}$-modules on $\mathcal{V}ar_{\bar{K}}$. Its cohomology $H^b \mathcal{G}^a$ is h-sheafification of the presheaf $(V, \bar{V}) \mapsto H^b(\bar{V}, \Omega^a_{\langle V, \bar{V} \rangle})$ on $\mathcal{V}ar^{\mathrm{ss}}_{\bar{K}}$. Our $I.$ is a filtration on the presheaf $(V, \bar{V}) \mapsto \operatorname{gr}^m_F R\Gamma^\natural_{\mathrm{dR}}(V, \bar{V})$; passing to h-sheafification, we get a finite filtration $I.$ on $\operatorname{gr}^m_F \mathcal{A}^\natural_{\mathrm{dR}}$ with span $[0, m]$ and

$$\operatorname{gr}^I_a \operatorname{gr}^m_F \mathcal{A}^\natural_{\mathrm{dR}} = \mathcal{G}^a \otimes^L_{O_{\bar{K}}} \operatorname{gr}^{m-a}_F \mathrm{A}_{\mathrm{dR}}[-a]. \tag{4.2.2}$$

[15] We replace one $t_a \in \mathcal{O}(\bar{U}_{O_{K'}})$ by $t_a \pi^e_{K'}/\pi_{K_U}$.

[16] I am grateful to Luc Illusie for the proof.

Notice that the bottom cohomology H^0 of the bottom term $I_0 = \mathrm{gr}^I_0$ is the constant sheaf $O_{\bar{K}}$ and $\mathcal{C}one(\mathrm{gr}^m_F \mathrm{A}_{\mathrm{dR}} \to \mathrm{gr}^m_F \mathcal{A}^{\natural}_{\mathrm{dR}}) = \mathrm{gr}^m_F \mathcal{A}^{\natural}_{\mathrm{dR}}/H^0 I_0$. Therefore, by (4.2.2), the Poincaré lemma follows from the next assertion:

Theorem. *The cohomology $H^b\mathcal{G}^a$ are h-sheaves of $\mathbb{Q}$- (hence $\bar{K}$-) vector spaces for $(a, b) \neq (0, 0)$.*

Remark. The p-divisiblity of $H^b\mathcal{G}^0$, $b \neq 0$, was first proved by Bhatt [Bh1, 8.0.1].

Exercise. Consider a presheaf $(V, \bar{V}) \mapsto R\Gamma(\bar{V}, L\Omega^{\cdot}_{(\hat{V},\bar{V})/(\bar{K},O_{\bar{K}})})$; let $\mathcal{A}^{naive}_{\mathrm{dR}}$ be its h-sheafification. One has an evident map $\mathcal{C}one(F^1\mathrm{A}_{\mathrm{dR}} \to \mathcal{A}^{\natural}_{\mathrm{dR}}) \to \mathcal{A}^{naive}_{\mathrm{dR}}$. Show that the theorem implies that it is a filtered quasi-isomorphism; i.e., the triangle $F^1\mathrm{A}_{\mathrm{dR}} \to \mathcal{A}^{\natural}_{\mathrm{dR}} \to \mathcal{A}^{naive}_{\mathrm{dR}}$ is exact in the filtered derived category of h-sheaves.

4.3. We deduce the above theorem from a more concrete assertion. As in §4.1, for an ss-pair $(U, \bar{U})$ over K we have the locally free $\mathcal{O}_{\bar{U}}$-module of log differentials $\Omega_{\langle U,\bar{U}\rangle} := \Omega_{(U,\bar{U})/(K_U,O_{K_U})}$ and its exterior powers $\Omega^a_{\langle U,\bar{U}\rangle} := \Lambda^a\Omega_{\langle U,\bar{U}\rangle}$.

Let $f : (U', \bar{U}') \to (U, \bar{U})$ be a map in $\mathcal{V}ar^{\mathrm{ss}}_K$ or $\mathcal{V}ar^{\mathrm{ss}}_{\bar{K}}$. We say that f is *(Hodge) p-negligible* if the morphisms $(\tau_{>0}R\Gamma(\bar{U}, \mathcal{O}_{\bar{U}})) \otimes^L \mathbb{Z}/p \to (\tau_{>0}R\Gamma(\bar{U}', \mathcal{O}_{\bar{U}'})) \otimes^L \mathbb{Z}/p$ and $R\Gamma(\bar{U}, \Omega^a_{\langle U,\bar{U}\rangle}) \otimes^L \mathbb{Z}/p \to R\Gamma(\bar{U}', \Omega^a_{\langle U',\bar{U}'\rangle}) \otimes^L \mathbb{Z}/p$, $a > 0$, in $D^b(O_{K_U}/p)$, resp. $D^b(O_{\bar{K}}/p)$, vanish.

Remark. For $(U, \bar{U}) \in \mathcal{V}ar^{\mathrm{ss}}_K$ and a point $K_U \to \bar{K}$, one has $R\Gamma(\bar{U}_{O_{\bar{K}}}, \Omega^a_{\langle U_{\bar{K}},\bar{U}_{O_{\bar{K}}}\rangle}) = R\Gamma(\bar{U}, \Omega^a_{\langle U,\bar{U}\rangle}) \otimes^L_{O_K} O_{\bar{K}}$. Therefore the base change functor $\mathcal{V}ar^{\mathrm{ss}}_K \to \mathcal{V}ar^{\mathrm{ss}}_{\bar{K}}$ (see (2.2.2)) preserves p-negligible maps.

Theorem. *Every $U \in \mathcal{V}ar^{\mathrm{ss}}_K$ admits a p-negligible h-covering. Ditto for $\bar{K}$-pairs.*

The theorem implies the one in §4.2: Indeed, the $\bar{K}$-assertion shows that one has $(\tau_{>0}\mathcal{G}^0) \otimes^L \mathbb{Z}/p = 0$ and $\mathcal{G}^a \otimes^L \mathbb{Z}/p = 0$ for $a > 0$; since for a complex $\mathcal{G}$ the multiplication by p on $H^{\cdot}\mathcal{G}$ is invertible if and only if $\mathcal{G} \otimes^L \mathbb{Z}/p = 0$, we are done. Thus it yields the Poincaré lemma.

The above remark shows that the K-version of the theorem implies the $\bar{K}$-one. The proof of the K-version takes the rest of the section.

4.4. *For the rest of §4, "pair" means "arithmetic K-pair" (see §2.2).* We need further input from de Jong. A morphism $f : (C, \bar{C}) \to (S, \bar{S})$ of pairs is said to be a *family of pointed curves* (over $(S, \bar{S})$) if the map $\bar{C}_S := f^{-1}(S) \to S$ is smooth of relative dimension 1 with irreducible geometric fibers, and $D_{fS} := \bar{C}_S \smallsetminus C$, viewed as a reduced scheme, is étale over S. Such an f is *semi-stable* if, in addition, $\bar{C}/\bar{S}$ is a semi-stable family of curves, and the closure D_f of D_{fS} in $\bar{C}$ (the *horizontal divisor*), viewed as a reduced scheme, is étale over $\bar{S}$ and intersects each fiber of f at smooth points. A section $e : (S, \bar{S}) \to (C, \bar{C})$ of f is said to be *nice* if $e(\bar{S})$ intersects fibers of f at smooth points and $D_f \cap e(\bar{S}) = \emptyset$. Families of pointed curves over $(S, \bar{S})$ form a category $\mathcal{C}_{(S,\bar{S})}$ in the obvious manner, and a morphism of bases $\psi : (S', \bar{S}') \to (S, \bar{S})$ yields an evident pullback functor $\mathcal{C}_{(S,\bar{S})} \to \mathcal{C}_{(S',\bar{S}')}$ which preserves semi-stable families. A morphism $f' \to f$ in $\mathcal{C}_{(S,\bar{S})}$ is called an *alteration* if $(C', \bar{C}')$ is an alteration of $(C, \bar{C})$; it is a *semi-stable alteration* (of f) if, in addition, f' is semi-stable.

Theorem. *(a) Any family $f : (C, \bar{C}) \to (S, \bar{S})$ of pointed curves with $f : \bar{C} \to \bar{S}$ projective admits a semi-stable alteration f' h-locally over $(S, \bar{S})$.*

(b) One can find f' as above which has a nice section e. Moreover, for a given closed subscheme $P \subset \bar{C}$ such that $f(P) = \bar{S}$ and $P \cap \bar{C}_S \subset C$, one can find e such that the map $\bar{C}' \to \bar{C}$ sends $e(\bar{S})$ to P.
(c) For any semi-stable family of pointed curves $f : (C, \bar{C}) \to (S, \bar{S})$ with $(S, \bar{S})$ a strict ss-pair, there exists a semi-stable alteration $m : (C, \tilde{C}) \to (C, \bar{C})$ of f with $m|_C = id_C$ such that $m : \tilde{C} \to \bar{C}$ is an isomorphism over smooth points of f and $(C, \tilde{C})$ is an ss-pair.

Proof. (c) is [dJ1, 3.6]. (a) follows from [dJ2, 2.4 (i),(ii)] except that de Jong does not care to control the domain of smoothness of the semi-stable alteration of f. A miniscule modification of his argument permits us to do this; see Appendix 1. Alternatively, (a) follows directly from a far more precise result of Temkin [T, 1.5].[17]

Let us check (b). Every pair has a canonical alteration by the union of normalizations of its irreducible components, so we assume all the way that $\bar{S}$ is normal and irreducible. Since P as in (b) exists h-locally on $(S, \bar{S})$,[18] we can assume it is given. Replacing $(S, \bar{S})$ by its alteration (P_S, P), we get a section e of f with image in P. Set $C^\flat := C \smallsetminus e(S)$. Then $(C^\flat, \bar{C}) \to (S, \bar{S})$ is a family of pointed curves; let $f'^\flat : (C'^\flat, \bar{C}') \to (S, \bar{S})$ be its semi-stable alteration as in (a). Let D_e be the closure in $\bar{C}'$ of the preimage D_{eS} of $e(S)$. Then D_e is an étale covering of $\bar{S}$.[19] Let C' be the preimage of $C \subset \bar{C}$ in $\bar{C}'$; then $(C', \bar{C}') \to (S, \bar{S})$ is a semi-stable alteration of $(C, \bar{C}) \to (S, \bar{S})$. Replacing $(S, \bar{S})$ by its alteration (D_{eS}, D_e), we get a nice section of $(C', \bar{C}')$ which sits over e, hence over P. $\square$

Remark. In (c), every nice section of $(C, \bar{C})$ lifts to a nice section of $(C, \tilde{C})$.

Corollary. *Any pair $(U, \bar{U})$ has an h-covering by ss-pairs $(C, \bar{C})$, $\dim C = \dim U$, for which there is a semi-stable family of pointed curves $f : (C, \bar{C}) \to (S, \bar{S})$ with a nice section such that $(S, \bar{S})$ is an ss-pair and C is affine over S (i.e., $f(D_f) = \bar{S}$).*

Proof. It suffices to find an h-covering of $(U, \bar{U})$ by pairs $(C, \bar{C})$ with $\dim C = \dim U$ for which there exists a family of pointed curves $f : (C, \bar{C}) \to (S, \bar{S})$ with C affine over S and projective $\bar{S}$, $\bar{C}$. The theorem transforms it then, with an input from Remark (i) in §2.5 to preserve the dimension and de Jong's theorem in §2.3 to alter $(S, \bar{S})$ from (b) into a strict ss-pair, into a datum with all promised properties.

By de Jong's theorem in §2.3, we can assume that $(U, \bar{U})$ is an ss-pair and $\bar{U}$ is projective and irreducible;[20] set $d = \dim U$. Pick any closed point $u \in U$. It suffices to find an open neighborhood $U' \subset U$ of U, an alteration $(C, \bar{C})$ of $(U', \bar{U})$, and a family of curves $f : (C, \bar{C}) \to (S, \bar{S})$ such that $f(D_f) = \bar{S}$.

Embed $\bar{U}$ into a projective space $\mathbb{P}^N_{O_K}$. By Bertini, there is a plane $H \subset \mathbb{P}^N$ defined over K of codimension d such that $u \notin H$, H intersects $\bar{U}_K$ transversally, $H \cap \bar{U}_K \subset U$, and the codimension $d-1$ plane which contains H and u is transversal to $\bar{U}_K$ and $\bar{U}_K \smallsetminus U$. Let $m : \bar{C} \to \bar{U}$ be the blowup at $\bar{U} \cap H_{O_K}$, $p : \bar{C} \to \mathbb{P}^{d-1}_{O_K}$ be the projection defined by H, and $\bar{C} \xrightarrow{f} \bar{S} \to \mathbb{P}^{d-1}_{O_K}$ be the Stein factorization of p (so $\bar{S} = \operatorname{Spec} p_* \mathcal{O}_{\bar{C}}$). Let $D \subset \bar{C}$ be the union of $m^{-1}(\bar{U} \smallsetminus U)$ and the exceptional divisor (viewed as a reduced scheme), and $S \subset \bar{S}_K$ be the maximal open subset

[17] To bring our datum to Temkin's setting, one flattens $\bar{C}/\bar{S}$ and $D_f/\bar{S}$ using [RG, 5.2.2] and replaces $\bar{S}$ by its normalization.

[18] Indeed, one can cover S by open subsets $\{S_\alpha\}$ such that P exists for $(C_{S_\alpha}, \bar{C}) \to (S_\alpha, \bar{S})$.

[19] Since $D_{f'^\flat}$ is étale over $\bar{S}$, D_{eS} is open and closed in $D_{f'^\flat S}$, $f'^\flat(D_{eS}) = S$, and $\bar{S}$ is normal.

[20] We only need that $\bar{U}$ is projective and normal, and that U is smooth.

over which f is smooth and $f|_D$ is étale. Set $C := f^{-1}(S) \smallsetminus D$ and $U' := m(C)$; notice that $m|_C : C \xrightarrow{\sim} U'$. Then U', $(C, \bar{C})$, f satisfy the promised properties (one has $f(D_f) = \bar{S}$ since D_f contains the exceptional divisor), q.e.d. □

4.5. Let us return to the proof of the theorem in §4.3. We use induction by $\dim U$. By the corollary in §4.4, we can replace $(U, \bar{U})$ by $(C, \bar{C})$ as in loc. cit., so we have $f : (C, \bar{C}) \to (S, \bar{S})$ with a nice section e and C affine over S. Notice that $(C, \bar{C})$ is log smooth over $(S, \bar{S})$ and the line bundle $\omega_f := \Omega_{(C,\bar{C})/(S,\bar{S})}$ equals $f^!(\mathcal{O}_{\bar{S}})[-1] \otimes \mathcal{O}_{\bar{C}}(D_f)$.

Key lemma. *h-locally over $(S, \bar{S})$, one can find a semi-stable alteration $\phi : f' \to f$ together with a nice section e' that lifts e such that $(C', \bar{C}')$ is an ss-pair and the pullback maps $\phi^* : R^1 f_* \mathcal{O}_{\bar{C}} \to R^1 f'_* \mathcal{O}_{\bar{C}'}$, $f_* \omega_f \to f'_* \omega_{f'}$ are divisible by p.*[21]

For a proof, see §4.6. Assuming it for the moment, let us finish the proof of the theorem in §4.3. By Remark (i) in §2.5, we can assume that the h-localization of $(S, \bar{S})$ in the Key Lemma does not change $\dim S$. We will show that for some h-covering $(S', \bar{S}')$ of $(S, \bar{S})$ the composition $(C', \bar{C}')_{(S',\bar{S}')} \to (C', \bar{C}') \xrightarrow{\phi} (C, \bar{C})$ is p-negligible.

For any a consider the exact sequence

$$0 \to f^* \Omega^a_{\langle S,\bar{S}\rangle} \to \Omega^a_{\langle C,\bar{C}\rangle} \to (f^* \Omega^{a-1}_{\langle S,\bar{S}\rangle}) \otimes \omega_f \to 0. \tag{4.5.1}$$

The section e splits off $\Omega^a_{\langle S,\bar{S}\rangle} \hookrightarrow Rf_* \Omega^a_{\langle C,\bar{C}\rangle}$ as a direct summand whose complement is $\mathcal{C}one(\partial_C)$, where $\partial_C : \Omega^{a-1}_{\langle S,\bar{S}\rangle} \otimes f_* \omega_f \to \Omega^a_{\langle S,\bar{S}\rangle} \otimes R^1 f_* \mathcal{O}_{\bar{C}}$ is the boundary map for (4.5.1) (one has $R^1 f_* \omega_f = 0$ since $f(D_f) = \bar{S}$). There is a similar splitting in case of f' provided by e', and the map $\phi^* : Rf_* \Omega^a_{\langle C,\bar{C}\rangle} \to Rf'_* \Omega^a_{\langle C',\bar{C}'\rangle}$ is compatible with the direct sum decompositions. Now ϕ^* is divisible by p on the second summand: Indeed, the Key Lemma asserts that the morphism of two-term complexes $\phi^* : \mathcal{C}one(\partial_C) \to \mathcal{C}one(\partial_{C'})$ is divisible by p on each term; since these are morphisms of vector bundles on O_K-flat $\bar{S}$, our $p^{-1}\phi^*$ is uniquely defined and commutes with the differentials. Thus the map $\phi^* \otimes \mathrm{id}_{C_1} : \mathcal{C}one(\partial_C) \otimes C_1 \to \mathcal{C}one(\partial_{C'}) \otimes C_1$, where $C_1 := \mathcal{C}one(p : \mathbb{Z} \to \mathbb{Z})$, is homotopic to zero. Apply $R\Gamma(\bar{S}, \cdot)$ and use the induction assumption to treat the first summand $R\Gamma(\bar{S}, \Omega^a_{\langle S,\bar{S}\rangle})$; we are done.

4.6. *Proof of Key Lemma.* Consider the relative Picard $\bar{S}$-schemes $J := Pic^0(\bar{C}/\bar{S})$ and $J^\flat := Pic^0((\bar{C}, D_f)/\bar{S})$: the first scheme parametrizes line bundles $\mathcal{L}$ on $\bar{C}$ such that the restriction of $\mathcal{L}$ to the normalization of each irreducible component of any geometric fiber of f has degree 0; the second one parametrizes pairs $(\mathcal{L}, \gamma)$, where $\mathcal{L}$ is as above and γ is a trivialization of $\mathcal{L}|_{D_f}$. Since $(\bar{C}, D_f)$ is a semi-stable $\bar{S}$-family of d-pointed curves, $d := \deg(D_f)$, our J and $J^\flat$ are semi-abelian schemes (see [R]), and $J^\flat$ is an extension of J by a torus $\mathbb{G}_m^{D_f}/\mathbb{G}_m$.

Over S our $J^\flat$ is a generalized Jacobian; let $i : C \to J^\flat_S$ be the Abel-Jacobi map $i : C \to J^\flat_S$, $x \mapsto \mathcal{O}_{\bar{C}}(x - e)$. Let $C^\sim \to C$ be the i-pullback of the multiplication by p isogeny $p_{J^\flat} : J^\flat \to J^\flat$, and $\bar{C}^\sim \to \bar{C}$ be the normalization of $\bar{C}$ in $C^\sim$. Then $f^\sim : (C^\sim, \bar{C}^\sim) \to (S, \bar{S})$ is a family of pointed curves, which is an alteration of f. By the theorem in §4.4, h-locally over $(S, \bar{S})$ there is a semi-stable alteration f' of $f^\sim$

[21] As elements of the groups $\mathrm{Hom}_{\mathcal{O}_{\bar{S}}}(R^1 f_* \mathcal{O}_{\bar{C}}, R^1 f'_* \mathcal{O}_{\bar{C}'})$, $\mathrm{Hom}_{\mathcal{O}_{\bar{S}}}(f_* \omega_f, f'_* \omega_{f'})$.

with $(C', \bar{C}')$ semi-stable and equipped with a nice section e' which lies over e. Let us check that the alteration $\phi : f' \to f$ satisfies the conditions of the Key Lemma.

Set $J' := Pic^0(\bar{C}'/\bar{S})$ and $J'^\flat := Pic^0((\bar{C}', D_f)/\bar{S})$. We have the pullback morphisms $\phi^* : J \to J'$, $J^\flat \to J'^\flat$ of our semi-abelian schemes; over S we have the norm maps $\phi_{*S} : J'_S \to J_S$, $J'^\flat_S \to J^\flat_S$. Both are compatible with the projections $J^\flat \to J$, $J'^\flat \to J'$.

Since $\bar{S}$ is normal, for any semi-abelian $\bar{S}$-schemes A, B one has (see [FC, I 2.7])

$$\text{Hom}(A, B) \xrightarrow{\sim} \text{Hom}(A_S, B_S). \tag{4.6.1}$$

Thus ϕ_{*S} extends to morphisms $\phi_* : J' \to J$, $J'^\flat \to J^\flat$.

Notice that $R^1 f_* \mathcal{O}_{\bar{C}}$ is the Lie algebra of J, and, by Serre duality, $f_* \omega_f$ is dual to the Lie algebra of $J^\flat$; the same is true for f'. Our $\phi^* : R^1 f_* \mathcal{O}_{\bar{C}} \to R^1 f'_* \mathcal{O}_{\bar{C}'}$ is the Lie algebra map for $\phi^* : J \to J'$, and $\phi^* : f_* \omega_f \to f'_* \omega_{f'}$ is the map between the duals to the Lie algebras for $\phi_* : J'^\flat \to J^\flat$ (this is true over S, hence everywhere since $\bar{S}$ is O_K-flat).

By construction, $\phi_{*S} : J'^\flat_S \to J^\flat_S$ factors through $p_{J^\flat}$ over S; i.e., it is divisible by p in $\text{Hom}(J'^\flat_S, J^\flat_S)$. By (4.6.1), ϕ_* is divisible by p in $\text{Hom}(J'^\flat, J^\flat)$. Passing to Lie algebras, we see that $\phi^* : f_* \omega_f \to f'_* \omega_{f'}$ is divisible by p. Similarly, $\phi_{*S} : J'_S \to J_S$ is divisible by p. Notice that J_S, J'_S, being Jacobians of smooth projective curves, are self-dual abelian schemes, and $\phi^*_S : J_S \to J'_S$ is dual to ϕ_{*S}. Hence ϕ^*_S is divisible by p. So, by (4.6.1), $\phi^* : J \to J'$ is divisible by p. Passing to Lie algebras, we see that $\phi^* : R^1 f_* \mathcal{O}_{\bar{C}} \to R^1 f'_* \mathcal{O}_{\bar{C}'}$ is divisible by p, q.e.d. □

Appendix

Below is a proof of part (a) in the theorem from §4.4. It follows closely de Jong's argument from §§2–3 of [dJ2] with a minor change of the lemma below; we refer the reader to sections of [dJ2] for details.

(i) ([dJ2, 2.10]) One can assume that $\bar{S}$ is irreducible. By [RG, 5.2.2], there is a canonical modification of $\bar{S}$, which is projective and is an isomorphism over S, such that the strict transforms of $\bar{C}$ and D_f are flat over $\bar{S}$. Passing to them, we can assume that *all fibers of f have dimension* 1, *of $f|_{D_f}$ have dimension* 0.

(ii) ([dJ2, 3.4–3.5]) We say that a family of pointed curves is *good* if irreducible components of all its geometric fibers are curves whose normalization has genus ≥ 2. A good alteration is an alteration with a good source.

Lemma. *f admits a good alteration h-locally over $(S, \bar{S})$.*

Proof of Lemma. It suffices to find for any closed point s in S its open neighborhood $S_{(s)} \subset S$ and an alteration $(S'_{(s)}, \bar{S}')$ of $(S_{(s)}, \bar{S})$ such that the pullback of f to $(S'_{(s)}, \bar{S}')$ admits a good alteration. To do this, we define by induction a strictly increasing sequence of open subsets $\emptyset = V_0 \subset V_1 \subset \ldots$ of $\bar{S}$ and a sequence of finite extensions $F = F_0 \subset F_1 \subset \ldots$ of the field F of rational functions on $\bar{C}$ such that the normalization $\bar{C}_i$ of $\bar{C}$ in F_i has the following properties: (a) the map $\bar{C}_i \to \bar{S}$ is smooth at s, (b) the map $\pi_i : \bar{C}_i \to \bar{C}$ is étale at D_{fs}, (c) the normalizations of irreducible components of geometric fibers of $\bar{C}_i$ over points of V_i have genus ≥ 2. There is an open neighborhood $U_i \subset S$ of s over which $\bar{C}_i$ is smooth and π_i is étale at D_f. The induction stops when $V_n = \bar{S}$; set $S_{(s)} = U_n$. The promised good alteration is $(\pi_n^{-1}(C), \bar{C}_n)$ fibered over the normalization $(S'_{(s)}, \bar{S}')$ of $(S_{(s)}, \bar{S})$ in F_n.

Let x be the closed point of the closure of s in $\bar{S}$. The induction produces simultaneously an auxiliary sequence of finite subsets $T_0 \subset T_1 \subset \dots$ of closed points of $\bar{C}_x$; it starts with $T_0 :=$ the union of D_{fx} and the set of nonregular points of $\bar{C}_x$. The induction step: suppose we have V_{i-1}, F_{i-1}, T_{i-1}; let us construct V_i, F_i, T_i assuming that $V_{i-1} \neq \bar{S}$. Let y be any closed point in $\bar{S} \smallsetminus V_{i-1}$. Since $\bar{S}$ is projective, there is an affine open V which contains x and y. Let $\mathcal{L}$ be a very ample line bundle on $\bar{C}$. Replacing it by a sufficiently high power, we can assume that $\Gamma(\bar{C}_V, \mathcal{L}) \twoheadrightarrow \Gamma(\bar{C}_x, \mathcal{L}) \times \Gamma(\bar{C}_y, \mathcal{L})$. One can find a finite unramified extension[22] K' of K with residue field k' and two sections $\gamma_1, \gamma_2 \in \Gamma(\bar{C}_V, \mathcal{L}) \otimes_{O_K} O_{K'}$ which do not vanish at the generic points of irreducible components of $\bar{C}_x$, $\bar{C}_y$, such that $t = \gamma_1/\gamma_2$ yields generically étale finite maps $t_x : \bar{C}_x \otimes k' \to \mathbb{P}^1_x \otimes k'$, $t_y : \bar{C}_y \otimes k' \to \mathbb{P}^1_y \otimes k'$ étale over $\{0, 1, \infty\}$ and such that $t_x(T_{i-1}) \cap \{0, 1, \infty\} = \emptyset$. Pick $\ell \geq 5$ prime to p, and let F_i be an extension of F_{i-1} generated by K', μ_ℓ, $t^{1/\ell}$, and $(1-t)^{1/\ell}$. Let T_i be the union of T_{i-1} and the set of ramification points of t_x. The normalization $\bar{C}_i$ of $\bar{C}$ in F_i satisfies (a), (b), and satisfies (c) over some open set V_i which contains V_{i-1} and y. We are done. $\square$

(iii) It remains to show that *every good f admits a semi-stable alteration after a possible alteration of the base*. The genus of the generic fiber of f is ≥ 2, so $(\bar{C}_S, D_{fS})$ is a stable n-pointed curve over S (where n is the degree of D_{fS} over S). The Deligne-Mumford stack of stable n-pointed curves is proper, so, after replacing $(S, \bar{S})$ by an alteration, we can assume that $(\bar{C}_S, D_{fS})$ extends to a stable n-pointed curve $(\bar{C}', D_{f'})$ over $\bar{S}$ (see [dJ2, 3.8]). We have a semi-stable family of pointed curves $f' : (C', \bar{C}') \to (S, \bar{S})$, $C' := \bar{C}' \smallsetminus D_{f'}$. By [dJ2, 3.10], the goodness of f implies that, after a possible alteration of $\bar{S}$, the evident morphism $\bar{C}'_S \to \bar{C}_S$ extends to a morphism $f' \to f$, and we are done. $\square$

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